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Hagedorn inflation of D-branes

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ABSTRACT: We examine the cosmological effects of the Hagedorn phase in models where the observable universe is pictured as a D-brane. It is shown that, even in the absence of a cosmological constant, winding modes cause a negative 'pressure' that can drive brane inflation of various types including both power law and exponential. We also find regimes in which the cosmology is stable but oscillating (a bouncing universe) with the Hagedorn phase softening the singular behavior associated with the collapse.

KEYWORDS: D-branes, Cosmology of Theories beyond the SM, Physics of the Early Universe.

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Contents

1. Introduction

Inflation [1] is a beautiful solution to several difficult problems in cosmology; the horizon problem, the flatness problem, and the monopole problem. In field theory however, the standard way to obtain inflation is to add a positive cosmological constant (which has a negative pressure $p = -\rho$). This ingredient is, without doubt, the least attractive feature of standard inflation and it is generally extremely difficult to control its adverse effects (e.g. the graceful exit and moduli problems). It is worth asking therefore if there are other forms of matter that can have a negative pressure and hence give an accelerating scale factor. In this paper we introduce a candidate that actually has this property — open strings on D-branes at temperatures close to the string scale.

At sufficiently high temperatures and densities fundamental strings enter a curious 'long string' Hagedorn phase [2]–[7]. To date applications of this phase have been quite limited in string cosmology [3] because the thermodynamics is governed by the finite temperature partition function. A rigorous analysis therefore requires nothing less than solving the string system in a cosmological setting, a difficult problem that might at best be tractable only in a few special cases. Moreover, in order to understand the effect of macroscopic phenomena such as winding modes we need the microcanonical ensemble (as we shall discuss) — an ensemble that does not particularly lend itself to cosmological applications.

In this paper we show that, for certain systems, it is possible to bypass these technical difficulties by using a classical random walk picture to model the behaviour of the strings in cosmological backgrounds. The particular systems we will focus on are D-branes in the weak coupling limit [8]. In this limit one can separate the energy momentum tensor into two components; a localized component corresponding to the D-brane tension, and a diffuse component that spreads into the bulk corresponding to open string excitations of the brane. (At the risk of causing confusion we will often refer to the latter as a 'bulk' component.) We will in addition allow a bulk cosmological constant, although our focus in this paper will be on the cosmology when the combined effect of the brane and bulk cosmological constants is subdominant.

The random walk picture allows us a first glimpse at the cosmological effects of a primordial Hagedorn phase of open strings on branes and we find two interesting types of behaviour.

- The first we call *Hagedorn inflation*. We will show that the transverse 'bulk' components of a D-brane's energy-momentum tensor can be negative. If all of the transverse dimensions have winding modes, this negative 'pressure' causes the brane to power law inflate along its length with a scale factor that varies as $a \sim t^{4/3}$ even in the absence of a nett cosmological constant (as shown in eq. (5.2)). If there are transverse dimensions that are large (in the sense that the string modes are not space-filling in these directions), then we can find exponential inflation (as shown in eq. (5.4)).

- If there is a small but negative cosmological constant, the Universe can enter a stable but oscillating phase; i.e. a 'bouncing' universe. The nett effect of the Hagedorn phase is to soften the singular behaviour associated with the collapse. Such singularity smoothing is a familiar aspect of strings, but the nice feature here is that we find it in a purely perturbative regime.

We should at this point also emphasize a general observation that we make, namely that the diffuse stringy component can have a dominant effect on the cosmology even in the weak coupling limit. At first sight this may be somewhat surprising given that the intrinsic tension energy of a D-brane goes like $\rho_{\rm br} \sim 1/g_s \sim 1/\hat{\kappa}$ where g_s is the string coupling and $\hat{\kappa}$ is the effective gravitational coupling. However, we will see that the *cosmological* effects of the two components are proportional to $\hat{\kappa}^4 \rho_{\rm br}^2$ and $\hat{\kappa}^2 \rho$ for the brane and diffuse 'bulk' components respectively. Then the contribution of the brane component is proportional to $\hat{\kappa}^4 \rho_{\rm br}^2 \sim \hat{\kappa}^2$. Since $1/g_s \gtrsim \rho \gtrsim 1$ to be in the Hagedorn phase, if e.g. the transverse volumes are of order unity in string units, then the contribution of the bulk component is $\hat{\kappa} \geq \hat{\kappa}^2 \rho \geq \hat{\kappa}^2$. Hence the cosmological effect of the diffuse bulk component can be dominant when g_s becomes small.

We begin in sections 2 and 3 by deriving the energy-momentum tensor, the principal ingredient of Einstein's equations. Since the results can be understood rather intuitively, this part of the discussion is organized so that cosmologists (and indeed anybody else) can skip the bulk of sections 2 and 3 concerning string thermodynamics and proceed directly to the energy-momentum tensor which is summarized at the end of section 3.

The thermodynamic discussion of section 2 gives a detailed introduction to the behaviour of both type-I and type-IIA/B open strings on D-branes as calculated from the microcanonical ensemble in a flat background. In particular we discuss the importance of macroscopic modes such as winding modes. Much of this section is a collation of results from ref. [6]. We then reintroduce the classical random walk picture paying special attention to the meaning of quantities such as average string length.

In section 3 we use the thermodynamic results to calculate the energy momentum tensor $T_{\mu\nu}$ of the Hagedorn phase. $T_{\mu\nu}$ enters into the higher dimensional Einstein's equations and determines the cosmology, and in particular we show that open string winding modes gives negative transverse components. For convenience the results for $T_{\mu\nu}$ are summarized at the end of section 3 where we also discuss the heuristic interpretation of this negative pressure.

Armed with the energy-momentum tensor, we examine the resultant cosmology. In sections 4 and 5, we solve the equations of motion with various ansätze, and find the advertised inflationary behaviour as well as bouncing solutions with singularity smoothing behaviour.

At this point we will also make some important observations regarding the regions of validity of the programme outlined in this paper. It is clear that there is some difficulty about using a purely classical description of gravity to describe such extreme systems. An important tool in knowing when our approach is valid is the phase diagram that was derived in ref. [6]. This diagram tells us, for example, when the entropy of black holes or black branes dominates over the stringy degrees of freedom. The phase diagram is reconsidered in section 5, where we find that in order to avoid strong gravitational effects, any transverse dimensions are limited in size as one might intuitively expect.

We will, purely for definiteness, consider adiabatic systems in solving the evolution equations of the universe. Under the assumption of adiabaticity, the inflationary growth period drives down the temperature of the system; eventually the temperature drop causes the universe to leave the Hagedorn regime and inflation ends automatically. Consequently adiabatic systems are probably unable to provide enough e-folds of inflation unless the string coupling is extremely weak. In section 6 we therefore discuss how, in non-adiabatic systems, inflation can be sustained. We conclude in section 7.

Before continuing, we make one final clarifying remark about the purpose of this paper. We shall be considering different phases of cosmological behaviour under extreme thermodynamic conditions. Our aim is *not* to end up with the usual 4-dimensional Einstein gravity, since this would require considering the stabilization of the extra dimensions which is beyond the scope of this paper. We will see directly (as mentioned above) that the cosmological effects of the D-brane tension are subdominant at weak string coupling. However we will also always be implicitly assuming that the cosmological effects of any additional stabilizing potential are also subdominant.

2. The Hagedorn phase and random walks

This section presents some background thermodynamics needed to get $T_{\mu\nu}$ (which enters into Einstein's equations) in the Hagedorn regime. In section 2.1 we review the thermodynamic properties of D-branes in toroidal compactifications. These compactifications allow us to use the *microcanonical* ensemble, which is defined in terms of global parameters such as total energy and volume. The importance of the microcanonical ensemble is that it allows a rigorous understanding of the effect of winding modes. This understanding enables us in sections 2.2 and 2.3 to extend the analysis to more general universes of any shape; in particular, we can determine those cases in which thermodynamics and hence cosmology can be studied. Essentially, we will argue that thermodynamics only makes sense in a cosmological setting for those systems in which the canonical and microcanonical ensembles agree. Where this is not the case the systems are dominated by large scale fluctuations.

For the systems in which the two ensembles agree, we will derive a expression for the partition function based on a heuristic random walk argument. In particular this expression gives a geometric understanding of the partition function that allows us to discuss its validity in limits of high energy density, small volume, etc.

In section 3, we then use this partition function to find the energy momentum tensor, $T_{\mu\nu}$. Readers whose principal interest is cosmology may wish to read subsection 2.1 and then proceed directly to the summary of the results for $T_{\mu\nu}$ in section 3.4.

2.1 String thermodynamics and the Hagedorn phase

The Hagedorn phase arises in theories containing fundamental strings because they have a large number of internal degrees of freedom. Indeed, because of the existence of many oscillator modes, the density of states grows exponentially with energy ε, $\omega(\varepsilon) \sim \varepsilon^{-b} e^{\beta_H \varepsilon}$, where the inverse Hagedorn temperature β_H (where $\beta = 1/T$) and the exponent b depend on the particular theory in question (for example heterotic or type II) [2]. For type-I, -IIA, -IIB strings the numerical value of the inverse Hagedorn temperature is $\beta_H = 2\sqrt{2}\pi$ in string units. It is easy to see that thermodynamic quantities, such as the entropy, are liable to diverge at the Hagedorn temperature; obtaining the partition function Z with the canonical ensemble and multiplying by the usual Boltzmann factor $e^{-\beta\varepsilon}$, one finds an integral for the partition function (for large ε) $Z \propto \int d\varepsilon \varepsilon^{-b} e^{(\beta_H - \beta)\varepsilon}$ which diverges at $\beta = \beta_H$ for $b \leq 2$.

If an infinite amount of energy is required to reach T_H, then we say that the system is *limiting*. If not then the system is said to be *non-limiting*. Already the simple canonical ensemble above indicates that the Hagedorn temperature is a limiting temperature for the $b \leq 2$ cases [3]. The remaining systems seem to be non-limiting, and until recently it was thought that this might imply some kind of Hagedorn phase transition (drawing strongly on the analogy with the quark-hadron phase transition) to more fundamental degrees of freedom. However, in ref. [6] it was shown that all string systems are in fact limiting, including arbitrary Dp-branes (i.e. open strings attached to p dimensional defects).

The limiting/non-limiting question is a rather subtle one, and in order to resolve it one must first allow for large scale fluctuations by using the microcanonical ensemble (as pointed out in the early papers of Carlitz and Frautschi [2]) and second, retain full volume dependence (as pointed out for heterotic strings by Deo et al [4] and for open strings in ref. [6]). Once both of these factors are included, it becomes clear that in all cases the Hagedorn temperature is truly a limiting temperature rather than an indication of some sort of phase transition. Perhaps the best evidence for this is that the Hagedorn phase completes (by entropy matching) a phase diagram which includes other non-perturbative phases such as black holes. (There is a sense in which the entire Hagedorn phase can be thought of as a first-order phase transition from Yang-Mills/supergravity degrees of freedom to black-branes/black-holes. We return to this point later.)

Let us summarize the rigorous results from ref. [6] for open and closed strings in a toroidally compactified space. The most direct route to the thermodynamics is to evaluate the one loop partition function with the euclidean time coordinate, τ, compactified with radius β. However for the random walk discussion later, it is useful to begin with the density of states $\omega(\varepsilon)$ of a single string of energy ε from which the same results are obtained;

$$\omega(\varepsilon) = \begin{cases} \beta_H \dfrac{V_\parallel}{V_\perp} V_o \dfrac{e^{\beta_H \varepsilon}}{(\beta_H \varepsilon)^{\gamma_o+1}} & \text{open} \\ \beta_H V_c \dfrac{e^{\beta_H \varepsilon}}{(\beta_H \varepsilon)^{\gamma_c+1}} & \text{closed.} \end{cases} \tag{2.1}$$

In the above $V_\perp$ and $V_\parallel$ are the volumes transverse and perpendicular to the D-brane. For open strings,

$$\gamma_o = \frac{d_o}{2} - 1\,, \tag{2.2}$$

where d_o is the number of dimensions transverse to the brane in which there are *no windings* and V_o is the volume of this space (if there are windings in all dimensions, then $V_o = \mathcal{O}(1)$ in string units). Similarly, for closed strings,

$$\gamma_c = \frac{d_c}{2} \tag{2.3}$$

where d_c is the number of dimensions in which closed strings have *no windings* and, again, V_c is the volume of this space. Note that γ_o and γ_c are ε-*dependent* critical exponents, because winding modes are quenched or activated depending on the string energy. Below we shall use γ to stand for either γ_c or γ_o as appropriate.

We now collect the results obtained in ref. [6] in the *microcanonical ensemble* working in an approximation to the thermodynamic limit. The two main types of behaviour are the single-long-string or non-limiting **NL** behaviour (see discussion below where we discuss the consequences of placing 'non-limiting' systems in thermal contact with limiting systems), with entropy density

$$\sigma \equiv \frac{S}{V_\parallel} \tag{2.4}$$

of the form

$$\sigma_{\mathbf{NL}[\gamma]} \approx \beta_H \rho - \frac{1+\gamma}{V_\parallel} \log(\rho), \tag{2.5}$$

and the various types of 'limiting' behaviour $\mathbf{L}[\gamma]$

$$\sigma_{\mathbf{L}[\gamma]} \approx \begin{cases} \beta_H \rho + 2\sqrt{\dfrac{\rho}{V_\perp}} & \text{if } \gamma = -1 \\ \beta_H \rho + \dfrac{\gamma - 1}{\gamma} \rho^{\frac{\gamma}{\gamma-1}} & \text{if } \gamma = -\dfrac{1}{2}, \dfrac{1}{2} \\ \beta_H \rho + \log(\rho) & \text{if } \gamma = 0 \\ \beta_H \rho - e^{-\rho} & \text{if } \gamma = 1, \end{cases} \tag{2.6}$$

up to positive constants of $\mathcal{O}(\infty)$ in string units. Here

$$\rho \equiv E/V_{\|}\,, \tag{2.7}$$

where E is the total energy of the system. Note that the microcanonical ensemble results are expressed in terms of global parameters such as total energy which are valid in a toroidal compactification. We shall find that it is only possible to generalize the discussion for those systems in which the canonical and microcanonical results agree.

The $\mathbf{L}[-1]$ system is our 'standard' high energy regime. From eq. (2.2), we see that it corresponds to $d_o = 0$, so that all dimensions have windings and it is the system which is always reached provided that the volumes are finite and the energy density is high enough. These systems are equivalent to $D-1$ branes where D is the total number of space-time dimensions (in other words freely moving open strings). (Once there are many windings, we can T dualize the Dirichlet directions so that they become Neumann directions much smaller than the string scale — the winding modes become a spectrum of Kaluza-Klein modes indicating open strings which are energetic enough to probe all of the $D-1$ Neumann dimensions — even the small ones.)

There are two other sorts of behaviour, marginal limiting **ML** and weak limiting **WL** with entropy densities

$$\begin{aligned}\sigma_{\mathbf{ML}} &\approx \beta_H\,\rho - \rho^{2D-3}\,(V_{D-1})^{2D-4}\,e^{-\rho\,R^{D-3}}\\ \sigma_{\mathbf{WL}} &\approx \beta_c\,\rho + \frac{\beta_c\,f}{2}\,\rho - \frac{\beta_c^2\,V_{\|}\,f^2}{24}\,\rho^2\,,\end{aligned} \tag{2.8}$$

respectively, where $f = V_{\|}/V_{\perp}$.

Whether a system is limiting or not is a function of the global parameters of the system such as total energy, E, and volumes $V_{\perp}$, $V_{\|}$. For example, imagine increasing the energy of an $\mathbf{L}[-1/2]$ system. These systems are characteristic of an intermediate energy phase of open string systems possessing one large transverse dimension without windings. As the energy is increased windings will eventually be excited in this direction. If the transverse dimension is of order $R_{\perp}$, this happens at an energy threshold $E = R_{\perp}^2$, and for higher energies the thermodynamic behaviour changes to that of an $\mathbf{L}[-1]$ system, the universal high-energy regime.[1]

[1]Note that the decoupling of winding modes implied by these equations of state is universal and in particular independent of the zero-temperature vacuum contribution to the partition function which may or may not be finite as the transverse radius becomes infinite [9]. This is because in the finite temperature piece of the partition function the winding modes are always accompanied by a Boltzmann-like suppression factor. A related point is that as well as the above contributions there is a suppressed contribution to the entropy density of order $\sigma_{YM} \sim T^{D-2\gamma_o-3}$ corresponding to a Yang-Mills gas in the $D-2\gamma_o-3$ dimensions with windings and KK modes. In the euclidean approach this contribution comes from a UV cut-off in the Schwinger integral and represents a field-theoretic contribution which is subdominant for temperatures close to $1/\beta_H$ but again is universal.

The **NL** behaviour is so called because the temperature is higher than T_H as can easily be seen when we calculate the temperature of the subsystems from

$$\begin{aligned} S_i(E_i) &= \log \Omega_i(E_i) \\ T_i^{-1} &= \beta_i = \frac{\partial S_i}{\partial E_i}\,, \end{aligned} \tag{2.9}$$

where Ω_i is the microcanonical density of states in the subsystem i. The non-limiting systems formally obey

$$T_{\mathbf{L}} < T_H < T_{\mathbf{NL}}\,. \tag{2.10}$$

However it is important to realise that this simply means that the **NL** regimes are *transients* for a finite ten-dimensional volume. Equilibrium can never be achieved when they are in thermal contact with the surrounding (colder) limiting system of closed strings.

We can compare the microcanonical ensemble results reviewed in the previous two pages to the canonical ensemble by taking the thermodynamic limit (i.e. letting $V_{||}$ become infinite whilst keeping the density on the brane ρ constant). If we consider strings attached to a single brane embedded in $D = 10$ space time dimensions, then there is universal agreement between the canonical and microcanonical ensembles when there are $0 < d_\perp < 4$ transverse dimensions. In this case heading towards the thermodynamic limit quenches winding modes — the transverse dimensions can be thought of as effectively infinite in this limit. However when $d_\perp > 4$ (Dp-branes in ten dimensions with $p < 5$), modes as large as the transverse dimensions can be quenched if $V_{||} \ll R_\perp^2$, giving **NL** behaviour, or activated into a $\mathbf{L}[-1]$ system for $V_{||} \gtrsim \sqrt{V_\perp}$ (when the saddle point approximation is valid). Taking $V_{||} \to \infty$ with ρ fixed and $R_\perp$ constant leads to the onset of $\mathbf{L}[-1]$ behaviour in the thermodynamic limit. **WL** behaviour occurs in an intermediate region of parameter space where windings are being quenched in an $\mathbf{L}[-1]$ system and the usual saddle point approximation is invalid. Finally closed strings in the thermodynamic limit have a critical dimension $D = 3$, where D is the number of spacetime dimensions. For $D \leq 3$, we get standard canonical behaviour, $\mathbf{L}[(D-1)/2]$. On the other hand, for $D > 3$ winding modes generate 'marginally limiting' behaviour **ML**.

Some examples of different situations are shown (somewhat impressionistically) in figure 1 where the dimension dependence is evident. The entropic preference of strings for branes with higher dimensionality has the obvious interpretation that, when the strings are volume filling, the larger dimension branes 'cut across' more strings.

Note that in the microcanonical ensemble we always use the definition of temperature as derived from the microcanonical definition of entropy, $\beta = \partial_E S(E, V, N)$ where E is the energy of the total system. The discussion above illustrates the main reason for this rather convoluted set of definitions ($Z(\beta) \to \Omega \to S(E) \to \beta$); the

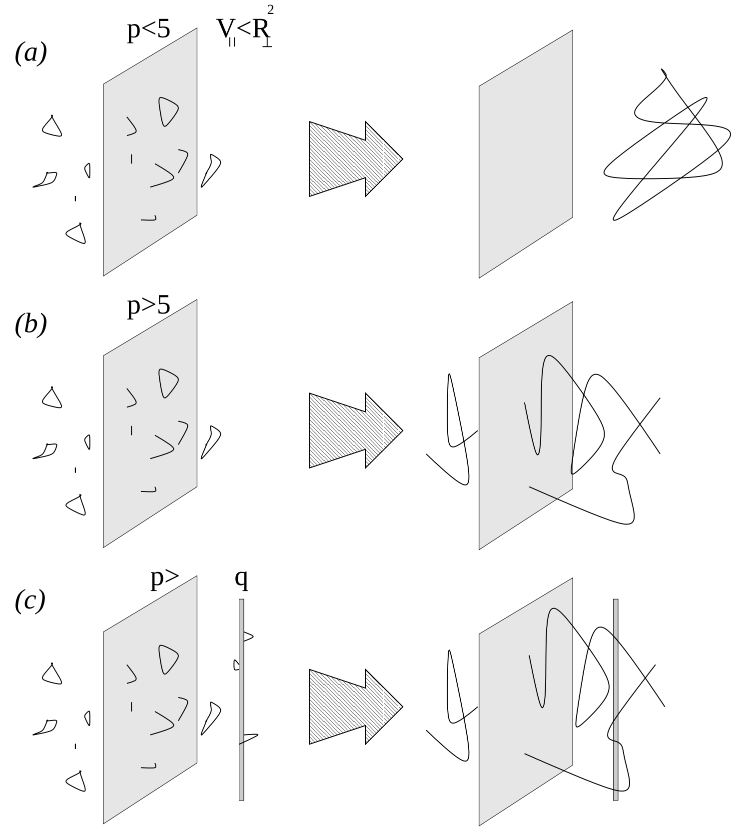

Figure 1: Examples of D-branes in thermal contact, showing (schematically) the direction of energy flow in order to achieve equilibrium. (a) A non-limiting (open string excitations of D-brane) plus marginal limiting (closed string) system. Energy is lost into the transverse volume until the energy density on the non-limiting system falls below the critical Hagedorn density. (b) A limiting (open string excitations of D-brane) plus marginal limiting (closed string) system. Energy flow is into open string winding modes as $T \to T_H$. (c) Limiting Dp brane plus limiting Dq brane plus marginal limiting (closed string) with $p > q$. Energy flows into open string windings on the Dp brane as $T \to T_H$.

effect of winding modes cannot fully be explored using the canonical ensemble. For example the canonical ensemble simply doesn't know about the **ML** and **WL** systems. The microcanonical ensemble also allows us to look at cases where parts of the system never come into equilibrium. This is true for instance for a D3-brane placed in a reservoir of closed strings. As we have seen, for large transverse volumes, a D3-brane formally has a temperature that is higher than T_H and so any thermal energy that it may have quickly evaporates into the surrounding bath of closed strings. Only the microcanonical ensemble allows one to compare the temperatures of *different* subsystems. Indeed the failure to do so leads to the false conclusion that the system *as a whole* might be non-limiting. Moreover, strictly speaking the thermodynamic limit (i.e. the assumption on which the canonical ensemble rests) does not actually exist for open strings. The phase diagram derived in ref. [6] tells us that when trying to take the thermodynamic limit one inevitably encounters a black hole phase. Thus it makes more sense to define the system with global quantities such as total energy, and work in finite volumes taking an approximation to the naive thermodynamic limit if possible.

2.2 The random walk interpretation

Despite its many advantages the microcanonical ensemble has one distinct disadvantage; in many cosmological backgrounds global properties such as total energy cannot be defined. We therefore need to be able to model these aspects of the thermodynamics with a more flexible approach. The random walk interpretation provides just such flexibility. For example, we can easily understand the **NL/L** cross over if we interpret a string of energy ε as a random walk of length $\sim \varepsilon$ in string units. The size of such a random walk is $\sqrt{\varepsilon}$ so that, roughly speaking, to have modes of size $R_\perp$ we require that the total energy $E = \rho V_{||} \sim V_{||} \sim R_\perp^2$. In this classical picture the **NL/L** cross over occurs precisely when there are modes as large as the transverse volume.[2] One helpful feature of the random walk interpretation is that we no longer have to bother about the precise topological properties of the space, merely its size in string units. On the other hand what we have gained from the rigorous analysis of the microcanonical ensemble is confidence in our understanding of which systems are stable, when one can use naive *canonical* ensemble results (i.e. as an approximation to the thermodynamic limit when the systems are limiting) and when the random walk interpretation is valid.

An analysis of random walks was the approach taken by Lowe and Thorlacius for closed strings and Lee and Thorlacius for open strings attached to D-branes [5]. These authors studied the Boltzmann equations for the average number of interacting strings sections of different lengths. The equilibrium solutions for closed string loops of length ℓ for example show that the (single string) density of states must be of the form $\omega(\varepsilon) \sim \frac{1}{\varepsilon} e^{\beta_H \varepsilon}$, where $\varepsilon = \sigma\ell$ and the temperature is related to the average total length $\bar{L}$ of string in the ensemble by

$$\beta_H = \beta - \frac{1}{\bar{L}}\,. \tag{2.11}$$

This supports the classical interpretation of the string as a random walk with, $\varepsilon = \sigma\ell$, where σ is the string tension encoding information about, for example, the step-size. In ref. [6] it was shown that such a random walk interpretation also accounts for the volume dependence as follows. First consider the distribution function $\omega(\varepsilon)$

[2] We should caution against taking this picture too literally although, as we are about to see, it does better than might be expected in explaining the thermodynamics. In particular the way to measure the 'size' of the string is to put it in a box with a certain energy and see at what radius it begins to excite winding modes. This should be borne in mind in order to avoid getting into circular arguments.

for closed strings in D large space-time dimensions. The energy ε of the string is proportional to the length of the random walk. The number of walks with a fixed starting point and a given length ε grows exponentially as $\exp(\beta_H \varepsilon)$. Since the walk must be closed, this overcounts by a factor of the volume of the walk, which we shall denote by

$$V(\text{walk}) = W\,. \tag{2.12}$$

Finally, there is a factor of V_{D-1} from the translational zero mode, and a factor of $1/\varepsilon$ because any point in the closed string can be a starting point. The end result is

$$\omega(\varepsilon)_{\text{closed}} \sim V_{D-1} \cdot \frac{1}{\varepsilon} \cdot \frac{e^{\beta_H \varepsilon}}{W}\,. \tag{2.13}$$

Now, the volume of the walk is proportional to $\varepsilon^{(D-1)/2}$ if it is well-contained in the volume ($R \gg \sqrt{\varepsilon}$), or roughly V_{D-1} if it is space-filling ($R \ll \sqrt{\varepsilon}$). From here we get the standard result [3, 4]. We have

$$\omega(\varepsilon)_{\text{closed}} \sim V_{D-1} \frac{e^{\beta_H \varepsilon}}{\varepsilon^{(D+1)/2}} \tag{2.14}$$

in D effectively non-compact space-time dimensions, and

$$\omega(\varepsilon)_{\text{closed}} \sim \frac{e^{\beta_H \varepsilon}}{\varepsilon} \tag{2.15}$$

in an effectively compact space. Note that these results agree with those of eq. (2.1); for example to obtain the same result as eq. (2.14), we take $d_c = D-1$, the number of space dimensions.

We can customize this analysis for open strings attached to a brane by a slight modification of the combinatorics. (The more general case of intersecting p, q branes was discussed in ref. [6]). The leading exponential degeneracy of a random walk of length ε with a fixed starting point in say the Dp-brane is the same as for closed strings: $\exp(\beta_H \varepsilon)$. Fixing also the end-point at a *particular* point of the Dp-brane requires the factor $1/W$ to cancel the overcounting, just as in the closed string case. Now, both end-points move freely in the part of the brane occupied by the walk. This gives a further degeneracy factor

$$W_{\|}^2 \tag{2.16}$$

from the positions of the end-points. Finally, the overall translation of the walk in the excluded transverse volume gives a factor $V_{\|}/W_{\|}$. The final result is:

$$\omega(\varepsilon)_{\text{open}} \sim \frac{V_{\|}}{W_{\perp}} \exp(\beta_H \varepsilon)\,. \tag{2.17}$$

Thus we see the sensitivity of the density of states to the effective volume of the random walk in the transverse directions. If the walk is well-contained in these directions, ($R_\perp \gg \sqrt{\varepsilon}$), then we find $W_\perp \sim \varepsilon^{d_\perp/2}$ and

$$\omega(\varepsilon)_{\rm open} \sim \frac{V_{\|}}{\varepsilon^{d_\perp/2}} \exp(\beta_H\,\varepsilon) \qquad R_\perp \gg \sqrt{\varepsilon}\,. \tag{2.18}$$

On the other hand, if it is space-filling in the transverse directions ($R_\perp \ll \sqrt{\varepsilon}$), the transverse volume of the walk is just $W_\perp \sim V_\perp$ and we find

$$\omega(\varepsilon)_{\rm open} \sim \frac{V_{\|}}{V_\perp} \exp(\beta_H\,\varepsilon) \qquad R_\perp \ll \sqrt{\varepsilon}\,, \tag{2.19}$$

in agreement with eq. (2.1).

2.3 Random walks in a cosmological background

The above thermodynamics involved a euclidean metric. How can we adapt the results for a cosmological background? First we assumed that the density of states increased as $\exp(\beta_H\,\varepsilon)$. In a sense the parameter ε is now no longer the energy but becomes merely something to be integrated over. In the case of a non-trivial metric the most natural interpretation of ε is that it is the proper length of a string in the bulk and certainly we can always go to the local inertial frame in which a small portion of the string has the usual euclidean energy $\equiv$ length equivalence. In addition it is clear that the average number of strings of a given proper length is well defined and so the arguments of Lee, Lowe and Thorlacius then give us the correct distribution of proper string lengths in equilibrium. Furthermore eq. (2.11) then gives us a working definition of temperature in terms of average proper string lengths.

To put these arguments on a firmer footing we now derive one aspect of the strings' behaviour in a cosmological background that will be useful in the discussions that follow; namely that in a slowly varying background a classical string of proper length $\ell \sim \varepsilon$ occupies a volume of proper-size $\sqrt{\varepsilon}$.

Consider a classical string of proper length $\ell = \varepsilon/\sigma$ (where $\sigma \sim 1$ in string units) with one end point fixed in a $D-1$ dimensional space. From our observations above we would expect such a string to have a density of states $\omega(\varepsilon) \sim e^{\beta_H \varepsilon}$. The crucial point is that we can arbitrarily divide this string into N small strings each of which has one end free and one end that is fixed to the end of the previous string. Consequently the density of states of the large single string is the same as that of N small strings each of which has one end fixed and with energies obeying $\sum_i \varepsilon_i = \varepsilon$ where ε_i is the energy of each string in its local inertial frame. By choosing a sufficiently large N we may always use the flat space approximation to evaluate the density of states of each small string rigorously (with one end fixed and one end free),

$$\omega_i(\varepsilon_i) = e^{\beta_H \varepsilon_i}\,. \tag{2.20}$$

In particular we find the same β_H for all the strings. The total density of states is then given by

$$\omega(\varepsilon) = \int \prod_i (\omega_i d\varepsilon_i)\, \delta(\varepsilon - \sum_i \varepsilon_i) = e^{\beta_H \varepsilon}\,, \tag{2.21}$$

as required.

It is now clear how to find the region occupied by the string. We measure this by determining the *gyration*, defined as the average size of the fluctuations of the free end from the fixed end, measured along null geodesics passing through the latter. Thermodynamics in the local inertial frames indicates that each small string has a spherically symmetric gyration with a radius $\sqrt{\varepsilon_i}$. To find the gyration of the large string we must (since they are average fluctuations) add those of the subsystems in quadrature. Hence, regardless of which geodesic we choose, the combined system has a gyration $\sqrt{\sum_i \varepsilon_i} = \sqrt{\varepsilon}$.

Proceeding now to the volume dependence of $\omega(\varepsilon)$, we first make the usual quasi-equilibrium approximation that equilibrium is established much more quickly than any change in the metric so that the metric may be taken to be approximately constant in time when evaluating properties such as density. In order to simplify matters, we also make the (by now) familiar assumption that the metric is factorizable into parallel dimensions x and transverse ones y

$$ds^2 = -n^2 dt^2 + g_{||ij} dx^i dx^j + g_{\perp nm} dy^n dy^m\,, \tag{2.22}$$

with the brane lying at $y^n = 0$. We define parallel and total volumes as

$$\begin{aligned} V_{||}(y') &= \int dx dy\, \delta(y - y') \sqrt{g_{||}} \\ W_{||}(y') &= \int dx dy\, \delta(y - y') \sqrt{g_{||}} \eta(y, x) \\ V &= \int dx dy\, \sqrt{g_{||}} \sqrt{g_\perp} \\ W &= \int dx dy\, \sqrt{g_{||}} \sqrt{g_\perp} \eta(y, x) \end{aligned} \tag{2.23}$$

where η is a function that is one in the region of the random walk and zero elsewhere, and where $g_{||}$ and $g_\perp$ are the determinants of the metric in the brane dimensions and transverse dimensions respectively. Note that we are using x and y as shorthand for all the parallel and transverse coordinates. We also define an averaging over the extra dimension with an overbar,

$$\overline{O} = \frac{\int dy \sqrt{g_\perp} O(y)}{\int dy' \sqrt{g_\perp}}\,. \tag{2.24}$$

For notational convenience we treat $\sqrt{g_\perp(y)}$ as a 'y-dependent transverse volume',

$$V_\perp(y) = \sqrt{g_\perp(y)} \int dy'\,, \tag{2.25}$$

so that the actual transverse volume is written

$$\overline{V_\perp} = \int dy \sqrt{g_\perp(y)}\,. \tag{2.26}$$

Note that $V = \overline{V_{||} V_\perp}$. From the discussion above, we have $W_{||} = \varepsilon^{d_{||}/2}$ reflecting the fact that the string has typical size $\sqrt{\varepsilon}$. For example if we take a slice at $y = y'$ this should be

$$W_{||} = \varepsilon^{d_{||}/2} = \int dx dy \delta(y - y') \sqrt{g_{||}} \eta = \int dx \sqrt{g_{||}}(y') \eta(y') \tag{2.27}$$

so that η must compensate for $\sqrt{g_{||}}$ in order to make this volume independent of y'.

In order to examine the thermodynamics we now need to decide when we can use the microcanonical results. We first divide the parallel dimensions into small locally flat patches with $\sqrt{g_\perp}$ approximately independent of x. In each patch it is consistent to use the microcanonical results, provided *there are no long range fluctuations.* Thus we are generally prohibited from examining the **NL** systems since adjacent patches will not be in equilibrium.[3]

The random walk argument proceeds as before with one change. The factor that corrects for the translation of the walk in the excluded volume is not $V_{||}/W_{||}$ but rather V/W for the $\mathbf{L}[-1]$ system. In a toroidal compactification these are of course the same for the $\mathbf{L}[-1]$ case. Here however, we must take account of the fact that the walk is squeezed by the metric so that it may have more 'room' at one side of the compactified dimension than the other. This factor can also be written as

$$\frac{V}{W} = \frac{\overline{V_{||}}}{W_{||}}\,, \tag{2.28}$$

and we now find, for example, that the density of states in the $\mathbf{L}[-1]$ system is

$$\omega(\varepsilon) \sim \frac{\overline{V_{||}}}{\overline{V_\perp}} e^{\beta_H \varepsilon}\,, \tag{2.29}$$

i.e. the same expression as in the flat space case but with all volumes averaged over transverse dimensions as in eqs. (2.24) and (2.25).

It is now possible to find $\log Z(\beta, \overline{V_{||}}, \overline{V_\perp})$ from $\omega(\varepsilon)$ by integrating over ε with a Boltzmann weighting, which we do for the various sytems in the following section. Note that the β appearing in the resulting partition function is the random walk definition of 'inverse temperature' whose precise physical interpretation is given by eq. (2.11).

[3]Also note that by examining the thermodynamics in a small region we do not artificially quench Kaluza-Klein modes as long as the local patches are much larger than the string scale. This is not the case in the perpendicular directions however because of the presence of macroscopic winding modes. Hence in a local patch with average volume $\overline{V_{||}}$ we may define an energy density by for example $\rho = E/\overline{V_{||}}$ where E is the total energy in the patch. However, for the thermodynamics, we must always include the whole transverse volume $\overline{V_\perp}$ to avoid artifically *un*quenching winding modes.

3. Stress-energy tensor $T_{\mu\nu}$ in a bulk Hagedorn phase

We now use these thermodynamic results to find the bulk energy momentum tensor during the Hagedorn regime. We may find the energy momentum tensor from

$$\langle T^{\mu}_{\nu} \rangle = 2 \frac{g^{\mu\rho}}{\sqrt{g}} \frac{\delta \log Z(\beta, \overline{V_{\|}}, \overline{V_{\perp}})}{\delta g^{\rho\nu}} , \tag{3.1}$$

where β, $\overline{V_{\|}}$, $\overline{V_{\perp}}$ are given by eqs. (2.11), (2.23), (2.24) and (2.25). We will treat the functional derivative with respect to $g_{\mu\nu}$ in the following way. We assume that small changes in the metric correspond to making small changes in the volumes in $\log Z$, e.g. for a single extra dimension

$$\frac{\delta Z}{\delta g_{55}} = \int dx' \frac{\delta Z}{\delta \overline{V}_{\perp}(x')} \frac{\delta \overline{V}_{\perp}(x')}{\delta g_{55}} . \tag{3.2}$$

Then, in the case of only one extra dimension, we can write

$$\overline{V_{\perp}} = \int dy \sqrt{g_{55}} = \frac{\int d^5x \sqrt{g_{55}}}{\int d^4x} = \frac{1}{v\beta} \int d^5x \sqrt{g_{55}} , \tag{3.3}$$

and

$$\frac{\delta \overline{V_{\perp}}}{\delta g_{55}} = \frac{1}{2\sqrt{g_{55}} v\beta} . \tag{3.4}$$

Then from eqs. (3.2)–(3.4) we can determine the functional derivative in eq. (3.1). Our ansatz automatically means that $T_{05} = 0$ and hence $G_{05} = 0$; in other words we are not considering energy exchange between the brane and the bulk. (In general there might be energy flux between the two.)

We begin by evaluating the energy momentum tensor for our 'standard' case of $\mathbf{L}[-1]$ (windings in all transverse dimensions) appropriate to high energies and densities; the resulting $T_{\mu\nu}$ is presented in eq. (3.11). We then proceed to $\mathbf{L}[\gamma \neq -1]$ cases (in which windings are quenched in some of the transverse dimensions) and present the resulting $T_{\mu\nu}$ in eq. (3.28).

3.1 $T_{\mu\nu}$ at high energies and densities; $\mathbf{L}[-1]$

We now apply eq. (3.1) to the very high energy regime in which there are modes stretching across the whole space in all transverse dimensions. In the nomenclature of ref. [6] these are the limiting $\mathbf{L}[-1]$ systems since $d_o = 0$.

We first need the principal cosmological ingredient, the partition function. The expression for the density of states in the $\mathbf{L}[-1]$ system is

$$\omega(\varepsilon) \sim \frac{\overline{V_{\|}}}{\overline{V_{\perp}}} e^{\beta_H \varepsilon} , \tag{3.5}$$

which, when integrated with a Boltzmann weighting, gives

$$\log Z(\beta, \overline{V_{\|}}, \overline{V_{\perp}}) = 2\frac{\overline{V_{\|}}^2 \beta_H^2}{\overline{V_{\perp}}(\beta^2 - \beta_H^2)} + a_c \overline{V_{\|}} - \overline{V_{\|}}\rho_c \frac{(\beta^2 - \beta_H^2)}{2\beta_H}\,, \tag{3.6}$$

where a_c and ρ_c are a critical pressure and energy density (defined with reference to the brane dimensions, i.e. with dimensions $E_c/\overline{V_{\|}}$) which are of order unity in string units [6]. Here, subscript-c refers to critical quantities to remain in the Hagedorn phase. These are the successive terms in a saddle point approximation.

After doing the functional derivative we can replace β by using the saddle point result for the $\mathbf{L}[-1]$ system,

$$2\frac{\overline{V_{\|}}\beta_H^2}{\overline{V_{\perp}}(\beta^2 - \beta_H^2)} \approx \sqrt{\frac{\overline{V_{\|}}}{\overline{V_{\perp}}}\beta_H(\rho - \rho_c)}\,, \tag{3.7}$$

where as before, $\rho = E/\overline{V_{\|}}$. For the saddle point approximation to be valid we require [6]

$$(\rho - \rho_c) \gg R_{\perp}^4\,. \tag{3.8}$$

This is also the condition for the leading (first) term in $\log Z$ to dominate over the a_c term, and for the a_c term to dominate over the ρ_c term. We neglect higher order terms.

In order to write the resulting energy-momentum tensor, we define the parameters

$$\alpha_{\|} = \frac{\overline{V_{\|}}}{V_{\|}}\,, \qquad \alpha_{\perp} = \frac{\overline{V_{\perp}}}{V_{\perp}} \tag{3.9}$$

and a pressure

$$\hat{p}_{-1} = \frac{\sqrt{\rho}}{\overline{V_{\perp}}^{3/2}}\,. \tag{3.10}$$

In the following section we shall argue that $\alpha_{\|} \approx 1$ wherever the Hagedorn phase is the dominant phase. If the off-diagonal components of the metric are small, then the variation with respect to $g^{\mu\nu}$ gives the bulk $\langle \hat{T}_{\mu\nu} \rangle$ terms;

$$\begin{aligned} \hat{T}_0^0 &= -\hat{\rho} = -\frac{\rho\alpha_{\|}\alpha_{\perp}}{\overline{V_{\perp}}} \\ \hat{T}_i^i &= \hat{p}_{-1} \\ \hat{T}_m^m &= -\hat{p}_{-1}(2\alpha_{\|} - 1) \approx -\hat{p}_{-1}\,, \end{aligned} \tag{3.11}$$

where i labels the parallel x directions and m labels the transverse y directions. We have dropped the $\langle\ \rangle$ notation for the expectation value of the energy momentum tensor, but shall continue to use hats to signify bulk properties such as the bulk density above, $\hat{\rho}$.

It is instructive to consider the conservation equations derived from the Bianchi identity $T^{\mu\nu}_{;\mu} = 0$. For example, restricting ourselves to the often discussed scenario with one extra dimension, y, and metric of the form

$$ds^2 = -n^2 dt^2 + a^2 dx^2 + b^2 dy^2 \,, \tag{3.12}$$

the E-M conservation equations may then be written (we are using the same notation as in ref. [10])

$$\begin{aligned} \frac{d\hat{T}^0_0}{dt} + (\hat{T}^0_0 - \hat{T}^i_i) 3\frac{\dot{a}}{a} + (\hat{T}^0_0 - \hat{T}^5_5)\frac{\dot{b}}{b} = 0 \\ \frac{d\hat{T}^5_5}{dy} + \hat{T}^5_5 \left(\frac{n'}{n} + \frac{a'}{a}\right) + \frac{n'}{n}\hat{T}^0_0 - 3\frac{a'}{a}\hat{T}^i_i = 0 \,. \end{aligned} \tag{3.13}$$

We shall see shortly that $\alpha_{||}, \alpha_\perp \approx 1$ so the first equation is a conservation law for entropy;

$$\frac{d(\sigma\overline{V_{||}})}{dt}\left(1 + \sqrt{\frac{\overline{V_\perp}}{\rho}}\right) - 2\sqrt{\frac{\overline{V_\perp}}{\rho}} \cdot \frac{d}{dt}\left(\overline{V_{||}}\sqrt{\frac{\rho}{\overline{V_\perp}}}\right) \tag{3.14}$$

where

$$\sigma = \rho + 2\sqrt{\frac{\rho}{\overline{V_\perp}}} \,. \tag{3.15}$$

Recall from ref. [6] that the saddle point dominance condition is

$$\sqrt{\frac{\overline{V_\perp}}{\rho}} \ll \frac{1}{R_\perp^2} \ll 1 \,. \tag{3.16}$$

Hence to order $1/R_\perp^2$ we simply recover the expected entropy conservation law, $d(\sigma\overline{V_{||}})/dt = 0$. Note that the entropy is evenly distributed because of the dominant first term; indeed to first order $\hat{T}^0_0 = \hat{\sigma}(y)$ where $\hat{\sigma}$ is the local entropy density in the bulk.

3.2 $T_{\mu\nu}$ for the $\mathbf{L}[\gamma \neq -1]$ systems

As the energy density of an $\mathbf{L}[-1]$ system falls below certain energy thresholds, the modes that are sensitive to the size of the transverse dimensions (e.g. winding modes in toroidal compactifictions) become quenched. Once this happens, the compactness of the quenched directions is of no further consequence for the thermodynamics, which aquires a different critical exponent, γ, and can even temporarily become **NL** (although the name 'non-limiting' is probably misleading for the reasons discussed in the Introduction and in ref. [6]). If there are many transverse dimensions, and they are anisotropic, there may be a few energy thresholds and γ may assume several (increasing) values before the density drops below the Hagedorn density ρ_c and the system finally leaves the Hagedorn phase and enters the Yang-Mills phase.

In order to discuss the thermodynamics of these intermediate systems, we first introduce the function $\eta'(y)$ which gives support in the regions occupied by the random walking strings in the various transverse dimensions. (Note that $\eta(x,y)$ described the shape of a single string whereas $\eta'(y)$ is for all strings, so $\eta'(y) = 0$ indicates that strings do not extend this far into the bulk.)

We define an averaging over the region given support by η';

$$\overline{O}_{\eta'} = \frac{\int dy \eta' \sqrt{g_\perp} O(y)}{\int dy' \eta' \sqrt{g_\perp}} \,. \tag{3.17}$$

The single string density of states density of states now involves the ratio of the volumes

$$\begin{aligned} V_{\eta'} &= \int dx dy \eta' \sqrt{g_\perp} \sqrt{g_{||}} \\ W &= \int dx dy \eta' \eta \sqrt{g_\perp} \sqrt{g_{||}} \,, \end{aligned} \tag{3.18}$$

(where η is defined below eq. (2.23)) so that

$$\frac{V_{\eta'}}{W} = \frac{\overline{V_{||}}_{\eta'}}{W_{||}} \,. \tag{3.19}$$

The expression for the single string density of states becomes

$$\omega(\varepsilon) \sim \frac{\overline{V_{||}}_{\eta'}}{\overline{V_\perp}_{\eta'}} \frac{e^{\beta_H \varepsilon}}{\varepsilon^\gamma} \tag{3.20}$$

where in accord with our previous definition of η, we have assumed that the strings occupy a proper volume $\varepsilon^{d_0/2}$ in the d_0 dimensions where the modes do not fill the whole of space. As before we have $\gamma_0 = d_0/2 - 1$.

Defining

$$\overline{f} = \frac{\overline{V_{||}}_{\eta'}}{\overline{V_\perp}_{\eta'}} \tag{3.21}$$

we may now use the expressions for the singular part of $\log Z$ found in ref. [6];

$$\log Z_{\text{sing}} \sim \begin{cases} \Gamma(-\gamma) \overline{f} (\beta - \beta_c)^\gamma, & \gamma \notin \mathbb{Z}^+ \cup \{0\} \\ \dfrac{(-1)^{\gamma+1}}{\Gamma(\gamma+1)} \overline{f} (\beta - \beta_c)^\gamma \log(\beta - \beta_c) & \gamma \in \mathbb{Z}^+ \cup \{0\}. \end{cases} \tag{3.22}$$

In the microcanonical ensemble, these expressions gives us the same relations as in eq. (2.6) with ρ replaced by

$$\overline{\rho} = \frac{E}{\overline{V_{||}}_{\eta'}} \,, \tag{3.23}$$

whence, substituting the appropriate expression for $\overline{\rho}$ from eq. (2.6) for each γ, we find the following for the $\mathbf{L}[\gamma \neq -1]$ systems;

$$
\begin{aligned}
\hat{T}^0_0 &= -\eta' \hat{\rho} \\
\hat{T}^i_i &= \eta' \hat{p}_\gamma \\
\hat{T}^m_m &= \begin{cases} -\eta' \hat{p}_\gamma \left(2\alpha_{||} - 1\right) \approx -\eta' \hat{p}_\gamma & \text{windings} \\ 0 & \text{no windings,} \end{cases}
\end{aligned}
\tag{3.24}
$$

where

$$
\hat{p}_\gamma = \begin{cases} \dfrac{1}{\overline{V}_{\perp\eta'}^{-3/2}} \overline{\rho}^{\frac{\gamma}{\gamma-1}} & \text{if } \gamma = -\dfrac{1}{2}, \dfrac{1}{2} \\ \dfrac{1}{\overline{V}_{\perp\eta'}^{-3/2}} \log \overline{\rho} & \text{if } \gamma = 0 \\ \dfrac{1}{\overline{V}_{\perp\eta'}^{-3/2}} e^{-\overline{\rho}} & \text{if } \gamma = 1\,. \end{cases}
\tag{3.25}
$$

Because, as we see later, $\alpha_{||,\perp} \approx 1$, we shall henceforth drop the overbar notation.

3.3 NL[γ], WL and ML systems

For the **NL** systems, $\gamma > 1$ and the microcanonical calculation tells us that

$$
E = -\frac{(1+\gamma)}{\beta - \beta_c}\,. \tag{3.26}
$$

In these cases we have already argued that they cannot be in equilibrium with the remaining systems since the temperature that we formally derive from the microcanonical ensemble has $T > T_H$. Indeed eq. (2.11) also indicates the average total length of strings in the ensemble cannot be positive. The fact that there is no equilibrium solution to the Boltzmann equations is merely a different way of seeing that these systems are transient. Little more information can be gained from the random walk picture in this case. However it seems likely that, as in the flat space microcanonical calculation, these systems tend to lose energy to closed strings in a cosmological setting.

The **WL** systems correspond to open strings that do not satisfy the saddle point condition. This is a cross over region where winding modes are just beginning to be excited, and the corresponding specific heat is therefore small (hence 'weak limiting'). Unfortunately this case is also difficult to analyse because the microcanonical ensemble does not agree with the canonical result (indeed there *is* no canonical equivalent of these systems). Consequently it is difficult to find anything that we may interpret as $\langle T^\nu_\mu \rangle$ for these cases. The same is true for the **ML** systems which correspond to closed strings in the Hagedorn regime. Fortunately all of these systems have lower entropy than the limiting systems and can be neglected in the cosmology.

3.4 Summary and discussion of $T_{\mu\nu}$

We now summarize the results for the energy-momentum tensor. We first drop the overline notation of the previous section and simply redefine $V_\perp$ and $V_{||}$ to be the transverse and parallel volumes covariantly averaged over the region of the transverse dimensions covered by the strings. (See eqs. (2.23), (2.25) and (3.17).) We define an energy density ρ of strings

$$\rho = \frac{E}{V_{||}}\,. \tag{3.27}$$

For the limiting systems with critical exponent γ, $\mathbf{L}[\gamma]$, we find that the 'bulk' components of the energy momentum tensor are given by

$$\begin{aligned} \hat{T}^0_0 &= -\hat{\rho} \\ \hat{T}^i_i &= \hat{p}_\gamma \\ \hat{T}^m_m &\approx \begin{cases} -\hat{p}_\gamma & \text{transverse with windings} \\ 0 & \text{transverse without windings,} \end{cases} \end{aligned} \tag{3.28}$$

where $\hat{\rho} = \rho/V_\perp$ and

$$\hat{p}_\gamma = \begin{cases} \dfrac{1}{V_\perp^{3/2}}\rho^{\frac{\gamma}{\gamma-1}} & \text{if } \gamma = -1, -\dfrac{1}{2}, \dfrac{1}{2} \\ \dfrac{1}{V_\perp^{3/2}}\log\rho & \text{if } \gamma = 0 \\ \dfrac{1}{V_\perp^{3/2}}e^{-\rho} & if \gamma = 1\,, \end{cases} \tag{3.29}$$

wherever there are strings present, and zero otherwise. The approximation in the transverse components is valid when the parallel volume changes only by a small fraction over the transverse directions.

For the remaining **NL**, **WL**, **ML** systems we do not know how to consider the cosmology or indeed whether they have any meaning at all in a cosmological setting. In the **NL** case large scale fluctations make it impossible to have a stable macroscopic system. For the **WL** and **ML** systems, one cannot properly take into account the effect of winding modes. The fact that the canonical and microcanonical ensembles disagree in a toroidally compactified space tells us that they play a dominant role in the thermodynamics, but it is not possible to formulate the required microcanonical treatment in cosmology since well defined global parameters such as total energy are generally lacking.

As expected $\hat{T}^0_0$ resembles the local energy density of strings. The $\hat{T}^i_i$ represents a relatively small pressure coming from Kaluza-Klein modes in the Neumann directions and $\hat{T}^m_m$ is a *negative* pressure coming from winding modes in the Dirichlet directions. If we T-dualize the Dirichlet directions these 'winding modes' also become Kaluza-Klein modes in Neumann-directions and T^m_m becomes positive. Thus negative T^m_m

reflects the fact that we have T-dualized a dimension much smaller than the string scale thereby reversing the pressure. For this reason negative T^{μ}_{μ} is expected to be a general feature of space-filling excitations in transverse dimensions.

Those who are familiar with T-duality may suspect an apparent conflict with this reversal of pressure when we T-dualize the extra dimensions. However we stress that T-duality is maintained. Pressure is merely *defined* as the change in free energy with *increase* in volume. Thus since free energy is invariant under T-duality the pressure must reverse sign. On the other hand the cosmological consequences in the T-dual system must of course be the same. In particular gravitons do not propagate in the extra dimensions of the T-dualized system (i.e. their Kaluza-Klein modes are heavy) since the extra dimensions are much smaller than the string scale. However gravitational degrees of freedom come from the closed string sector in the bulk which has both Kaluza-Klein modes *and* winding modes in all directions. Hence in the T-dual system the light gravitational degrees of freedom in the extra dimensions are winding modes. The original transverse components of Einstein's equations (which are valid only in the approximation that the Dirichlet directions are much larger than the string scale) must in the T-dual system be replaced by the equations of motion derived from the action now expressed as a sum over gravitational *winding* sectors. The upshot is that T-duality dictates that the resulting equations have the same consequences for the brane cosmology as the original system.

4. Cosmological equations in $D = 5$

In the previous sections we obtained the bulk stress-energy tensor at temperatures close to the Hagedorn temperature for the most general limiting cases in arbitrary numbers of dimensions. We now proceed to a discussion of the cosmology for which we will for the most part consider a 'toy-model' system with the following restrictions.

The first restriction is that we implicitely be considering a D-brane configuration that has 3 large parallel dimensions (i.e. the 'observable universe') and only one transverse dimension y that supports winding modes. Such a configuration could possibly arise in a 5 dimensional intersection of D-branes. A more realistic case would probably include all $D = 10$ dimensions but the Einstein equations are significantly more complicated in this case and we leave their full examination to a later paper.

The second restriction we make is simply for definiteness; we assume adiabaticity when solving Einstein equations. We shall find in this section and in section 5 both power law and exponential inflation. *In principal*, as we will see in section 6, it is therefore possible to solve the horizon problem even with adiabaticity. *In practice* however it is not possible to introduce *a priori* the entire entropy of the observable universe (i.e. $S \sim 10^{88}$) onto a primordial brane of order the string scale without either destroying the brane or having an unreasonably small string coupling (i.e. $g_s \sim 10^{-88}$!). In section 6 therefore we shall discuss possible deviations from adiabaticity.

We use the metric of eq. (3.12),

$$ds^2 = -n^2 dt^2 + a^2 dx^2 + b^2 dy^2 \,, \tag{4.1}$$

which foliates the space into flat, homogeneous, and isotropic spatial 3-planes. Here $\mathbf{x} = x_1, x_2, x_3$ are the coordinates on the spatial 3-planes while y is the coordinate of the extra dimension. For simplicity, we make a further restriction by imposing Z_2 symmetry under $y \to -y$. Without any loss of generality we choose the 3-brane of the 'observable universe' to be fixed at $y = 0$. (Even if the brane is moving we can always reparameterize the theory so that the 3-brane is fixed in coordinate space, although in doing so we lose further freedom of gauge choice [12].) With our assumption of homegeneity we can associate a scale factor with the parallel dimensions $a(t, y)$ and one for the 'extra' transverse dimensions $b(t, y)$. We define $a_0(t(\tau)) = a(t, 0)$ as the scale factor describing the expansion of the 3-brane where $t(\tau) \equiv \int d\tau n(\tau, y = 0)$ is the proper time of a comoving observer.

Several authors [10]–[15] have presented the bulk Einstein equations but for completeness we briefly restate the results;

$$\hat{G}_{00} = 3\left\{\frac{\dot{a}}{a}\left(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}\right) - \frac{n^2}{b^2}\left[\frac{a''}{a}+\frac{a'}{a}\left(\frac{a'}{a}-\frac{b'}{b}\right)\right]\right\} = \hat{\kappa}^2\,\hat{T}_{00}\,, \tag{4.2}$$

$$\begin{aligned}\hat{G}_{ii} &= \frac{a^2}{b^2}\left\{\frac{a'}{a}\left(\frac{a'}{a}+2\frac{n'}{n}\right) - \frac{b'}{b}\left(\frac{n'}{n}+2\frac{a'}{a}\right) + 2\frac{a''}{a} + \frac{n''}{n}\right\} + \\ &\quad + \frac{a^2}{n^2}\left\{\frac{\dot{a}}{a}\left(-\frac{\dot{a}}{a}+2\frac{\dot{n}}{n}\right) - 2\frac{\ddot{a}}{a} + \frac{\dot{b}}{b}\left(-2\frac{\dot{a}}{a}+\frac{\dot{n}}{n}\right) - \frac{\ddot{b}}{b}\right\} = \hat{\kappa}^2\,\hat{T}_{ii}\,,\end{aligned} \tag{4.3}$$

$$\hat{G}_{05} = 3\left(\frac{n'}{n}\frac{\dot{a}}{a} + \frac{a'}{a}\frac{\dot{b}}{b} - \frac{\dot{a}'}{a}\right) = \hat{\kappa}^2\,\hat{T}_{05}\,, \tag{4.4}$$

$$\hat{G}_{55} = 3\left\{\frac{a'}{a}\left(\frac{a'}{a}+\frac{n'}{n}\right) - \frac{b^2}{n^2}\left[\frac{\dot{a}}{a}\left(\frac{\dot{a}}{a}-\frac{\dot{n}}{n}\right)+\frac{\ddot{a}}{a}\right]\right\} = \hat{\kappa}^2\,\hat{T}_{55}\,, \tag{4.5}$$

where we use the notation of ref. [10] in which

$$\hat{\kappa}^2 = 8\pi\hat{G} = \frac{8\pi}{M_5^3}\,, \tag{4.6}$$

where M_5 is the five-dimensional Planck mass and the dots and primes denote differentiation with respect to t and y, respectively. Note that, as stated earlier, our ansatz implies that $T_{05} = 0$ in the bulk.

The bulk equations only apply in an open region that does not include the boundary. It is the Israel conditions which connect the boundary and the bulk [10]-[15];

$$\sum_{\pm\text{faces}} (K_{\mu\nu} - K h_{\mu\nu}) = t_{\mu\nu}\,, \tag{4.7}$$

where $K_{\mu\nu}$ is the extrinsic curvature, the sum over faces is for each side of the boundary surface and we have defined

$$t_{\mu\nu} = \frac{2}{\sqrt{h}} \frac{\delta S_{\text{boundary}}}{\delta h^{\mu\nu}} \tag{4.8}$$

as the energy momentum tensor on the boundary. We will assume that this energy momentum tensor on the boundary can be written in a perfect fluid form. For example,

$$t_0^0 = \rho_{\text{br}} \tag{4.9}$$

and

$$t_1^1 = -p_{\text{br}} , \tag{4.10}$$

where ρ_{br} and p_{br} are the energy density and pressure, respectively, measured by a comoving observer.

We can now find the Israel conditions specific to the metric in eq. (3.12). Because we are considering a brane at a Z_2 symmetry fixed plane, from the sum over faces we get two identical terms, merely resulting in a factor of two. In a more general space-time, the Israel conditions would still hold, but one would have to explicitly add the contributions from the two sides of the boundary surfaces since these contributions would no longer be equal. With the Z_2 symmetry, the Israel condition becomes,

$$\begin{aligned} 3[a'/a]_0 &= -\hat{\kappa}^2 b_0 \rho_{\text{br}} \\ 3[n'/n]_0 &= \hat{\kappa}^2 b_0 (2\rho_{\text{br}} + 3p_{\text{br}}) \end{aligned} \tag{4.11}$$

The same result can be found by adding $\delta(y) t^\mu_\nu$ to T^μ_ν and equating the coefficients of $\delta(y)$ in Einstein's equations.

It is interesting at this point to consider the relative contributions of the stringy excitations and of the D-brane tension to the cosmology. In particular we, somewhat counterintuitively, find that the diffuse stringy component can have a dominant effect on the cosmology even in the weak coupling limit. To see this, first note that the effective gravitational coupling $\hat{\kappa}$ is given in terms of the string coupling g_s by

$$\frac{1}{\hat{\kappa}^2} \sim \frac{m_s^{D-2}\,\text{Vol}_{D-5}}{g_s^2} \sim M_5^3 . \tag{4.12}$$

where Vol_{D-5} is the volume of the compactification from $D = 10$ to 5 dimensions, m_s is the string scale and M_5 is the 5-dimensional Planck mass. The intrinsic tension of a D-brane is given by

$$T_{\text{D}p} = \frac{1}{(2\pi)^p g_s} \sim \frac{1}{\hat{\kappa}\sqrt{\text{Vol}_{D-5}}} \tag{4.13}$$

For convenience we will henceforth assume that $\text{Vol}_{D-5} \sim m_s^{-5}$ so that g_s and $\hat{\kappa}$ are of the same order of magnitude in string units. (This need not be the case if some of the other space dimensions are compactified with a radius much larger or smaller than the string scale.) *We will also henceforth work in string units so that $\hat{\kappa}$ and g_s are equivalent.*

Thus the intrinsic tension of the D-brane satisfies $\rho_{\rm br} \sim 1/\hat{\kappa}$, and in the small coupling limit one might expect it to dominate the cosmology. However, this is not the case. If we substitute the Israel matching conditions into the cosmological equations (4.2-4.5), we see that the contributions of the D-brane tension and of the stringy components in the cosmological equations are of order

$$\hat{\kappa}^4 \rho_{\rm br}^2 \qquad \text{and} \qquad \hat{\kappa}^2 \frac{\rho}{V_\perp}\,, \qquad \hat{\kappa}^2 \hat{p}_\gamma$$

respectively. When the density of the string gas is close to the critical Hagedorn density, $\rho_c \sim 1$ (the effective lower bound), all the terms are of order $\hat{\kappa}^2$ and are therefore comparable. Since the density of the string gas can be significantly larger than the lower bound of $\rho_c \sim 1$, clearly the bulk stringy contribution can be significantly larger than the brane contribution.

We can further see the dominance of the bulk stringy component in the weak coupling limit. All our work has assumed that

$$\rho \lesssim T_{{\rm D}p}\,. \tag{4.14}$$

If this condition is violated, the thermal energy of the brane is larger than its rest mass and one would expect our perturbative treatment of the D-brane to break down (see ref. [6] for possible outcomes.) (Note that this bound does not apply *directly* to orientifolds which are stuck at fixed points.) Since $T_{{\rm D}p} \propto 1/\hat{\kappa}$, we can see that the amount of thermal energy that can be loaded onto the brane increases as the coupling gets weaker. When this bound is saturated, the stringy contribution to the cosmological equations can be $\mathcal{O}(\hat{\kappa})$ while that from the intrinsic tension is $\mathcal{O}(\hat{\kappa}^2)$; thus weak coupling is advantageous for string dominance in the cosmology. This rather surprising fact will become clear in the examples of the next section.

5. Results: Behaviour of scale factors $a(t)$ and $b(t)$

In this section we present results for the behaviour of the scale factors a and b. As discussed above, we will for simplicity and definiteness mainly consider 5 dimensional, adiabatic systems. In section 6 we will consider how it may be possible to relax these restrictions.

Our results are obtained by solving Einstein's equations together with the constraints provided by the Israel conditions. We use the energy momentum tensor derived above appropriate to a primordial Hagedorn epoch. First we will consider the regime $\mathbf{L}[\gamma]$ under the assumption that the brane is energy/pressureless and that

$\dot{b} = 0$. We also discuss some constraints imposed by the stability of the extra dimension. We then catalogue more general types of behaviour that can occur for $\dot{b} \neq 0$ and discuss a semi-realistic example in which b is constrained.

5.1 Hagedorn inflation; $\Lambda_{\rm br} = \Lambda_{\rm bulk} = \dot{b} = 0$

Let us first impose $\dot{b} = 0$ simply as an external condition; i.e. the extra dimension is fixed in time. We also set the cosmological constants both on the brane and in the bulk to zero, $\Lambda_{\rm br} = \Lambda_{\rm bulk} = 0$. All other contributions to the energy momentum localized on the brane (for example massless Yang-Mills degrees of freedom or possibly an **NL** subsystem) have an entropy that is subdominant to the limiting bulk degrees of freedom as long as $\rho > \rho_c$ (where ρ_c is a critical density of order 1). For the moment we will therefore neglect them and set $a'(t,0) = n'(t,0) = 0$. In this discussion the brane at $y = 0$ is playing no role in determining the evolution of the cosmology; the scale factor a_0 changes purely as a result of the bulk equations. (We shall see shortly that consistency of the full solutions requires a second brane at $y = l$).

5.1.1 Solutions to the T_{55} equation

It is simple to solve the 55 equation for the scale factor $a(t,y)$ at the origin,

$$a(t,0) = a_0(t)\,, \tag{5.1}$$

for all the $\mathbf{L}[\gamma]$ systems.

(a) $\mathbf{L}[-1]$: in the $\mathbf{L}[-1]$ systems (the high energy case with windings in all transverse dimensions) we see that $T_5^5 \sim \sqrt{\rho} \sim a^{-3/2}$ (where, again, ρ is the local energy density measured with respect to the volume $V_{||}$) and hence,

$$a_0(t) \sim t^{4/3}\,, \tag{5.2}$$

i.e. *power law inflation.*

(b) $\mathbf{L}[0]$: we find that $\gamma = 0$ gives us a period of *exponential inflation.* To find this solution we write $a_0^2 = \exp F$ and under the present assumption of negligible brane density and pressure, eqs. (4.11) and (4.5) lead to

$$\ddot{F} + \dot{F}^2 = \frac{2\hat{\kappa}^2}{3V_\perp^{3/2}}\left(\log\rho(0) - \frac{3}{2}F\right) \tag{5.3}$$

which may easily be solved to give

$$\frac{a_0(t)}{a_0(0)} = \exp\left(-\frac{t^2\hat{\kappa}^2}{8V_\perp^{3/2}} + t\sqrt{\frac{\hat{\kappa}^2}{6V_\perp^{3/2}}\log(\rho(0)e^{3/4})}\right) \tag{5.4}$$

where $\rho(0)$ is the initial density at $t = 0$. Initially the second term in the exponent dominates and there is exponential inflation. This solution has an

regime	$\rho(\beta) = E/V_{\parallel}$	$T^m_m = \eta' V_o/V_\perp^2 \times \cdots$	$a(t)/a(0)$
$\mathbf{L}[-1]$	$\frac{1}{V_\perp}(\beta - \beta_H)^{-2}$	$\rho^{1/2}$	$t^{4/p}$
$\mathbf{L}[-1/2]$	$(\beta - \beta_H)^{-3/2}$	$\rho^{1/3}$	$t^{6/p}$
$\mathbf{L}[0]$	$(\beta - \beta_H)^{-1}$	$\log \rho$	$\exp(-Ct^2 + Dt)$
$\mathbf{L}[1/2]$	$(\beta - \beta_H)^{-1/2}$	ρ^{-1}	$t^{-2/p}$
$\mathbf{L}[1]$	$-\log(\beta - \beta_H)$	$e^{-\rho}$	const.

Table 1: Cosmological regimes for open strings in the Hagedorn phase with p large parallel dimensions. The constants C and D are given for a $p = 3$ (i.e. the 'observable universe' as a 3-brane) in eq. (5.4). Note inflationary expansion for $\mathbf{L}[\gamma \leq 0]$.

automatic end to inflation when $da_0(t)/dt \sim 0$, i.e. when the two terms in the exponent are roughly of the same magnitude. Hagedorn inflation ends at time $t \sim \hat{\kappa}^{-1}\sqrt{V_\perp^{3/2} \log \rho(0)}$ which corresponds to $\log(a_0^3(t)/a_0^3(0)) \approx \log \rho(0)$. During a period of adiabaticity, the entropy and energy density dilute as $1/a_0^3$ because the Hagedorn phase is always to a first approximation like pressureless matter ($\hat{p}_{-1} \ll \hat{\rho}$). Hence we find that the above condition for inflation to end happens when $\rho(t) \sim 1$ (in string units), just as the system is dropping out of the Hagedorn phase.

(c) Other systems: we summarize the energy momentum tensors and cosmological behavior of $a_0(t)$ for the remaining systems in table 1, where $p = 3$ in our 'toy-model'. In all these cases it should be noted that during a period of adiabaticity, the entropy density dilutes as $1/a_0^3$. The reason for the various different types of behaviour is of course due to T_5^5 which may drop off more slowly than $1/a_0^3$ and in the $\mathbf{L}[0]$ case drops only logarithmically with the expansion. Note inflationary (superluminal) expansion for $\mathbf{L}[\gamma \leq \mathbf{0}]$.

In adiabatic systems the Hagedorn regime and hence the inflationary behaviour we have found eventually come to an end. For a system to be in the Hagedorn regime requires an entropy density higher than the critical Hagedorn density (of order 1 in string units). Below this density the energy momentum tensor and hence the cosmology is governed by the massless relativistic Yang-Mills gas (the gas is present on the brane even in the Hagedorn regime but is subdominant). Thus there is no problem exiting from the inflationary behaviour. In fact the main issue is how long inflation can last. We return to this question later when we discuss how inflation can be sustained.

5.1.2 Solutions to the T_{00} and T_{ii} equations

We can get an approximate solution for the remaining equations which is valid in all the regions of interest as follows. For small y we write

$$\begin{aligned} a(t,y) &= a_0(t) + \frac{y^2}{2}a_2(t) + \cdots \\ n(t,y) &= 1 + \frac{y^2}{2}n_2(t) + \cdots \end{aligned} \tag{5.5}$$

where henceforth we normalize $n(t,0) = 1$. First, given our constraints above, the 05 equation is trivially satisfied. As before we solve the 55 equation with $\hat{p}_\gamma = V_\perp^{-3/2}\rho^{2/3q}$ and find

$$a_0(t) \approx At^q\,, \tag{5.6}$$

where

$$b_0^{3/2}A^{2/q} = \frac{\hat{\kappa}^2}{3q(2q-1)}\,. \tag{5.7}$$

Solving the 00 and ii equations with $T_0^0 = -\hat{\kappa}^2\rho/V_\perp = -\hat{\kappa}^2\hat{\rho}$ we find

$$\begin{aligned} a_2(t) &= b_0^2 a_0\left(\frac{q^2}{t^2} - \frac{\kappa^2\hat{\rho}}{3}\right) \\ n_2(t) &= b_0^2\left(\frac{q^2-2q}{t^2} + \frac{\kappa^2(2\hat{\rho}+3\hat{p}_\gamma)}{3}\right). \end{aligned} \tag{5.8}$$

5.1.3 Phase diagram constraints on the transverse dimension and $\alpha_{||} \approx 1$

One important aspect of the approximate solution is that it is valid in all Hagedorn dominated regions. Obviously the approximation requires that the y^2 terms are small. Looking at $a(t,y)$ we see that the condition is $a_2y^2 < a_0$ or

$$\hat{\kappa}^2\hat{\rho} < \frac{1}{R_\perp^2}\,. \tag{5.9}$$

(We remind the reader that all quantities are expressed in string units so that $\hat{\kappa} \equiv g_s$.) Recall from ref. [6] that the Hagedorn regime occupies a small-coupling region between the Yang Mills phase and the black brane phase. Thus if the energy density is too high, thermodynamics is dominated by black branes and our entire perturbative derivation ceases to be valid. In addition there is a 'holographic' constraint which is saturated when the entropy is equal to that of a black Dp-brane filling the entire transverse volume (for a review see ref. [16]);

$$\hat{\kappa}^2\hat{\rho}V_\perp \approx \hat{\kappa}^2\rho \ll V_\perp^{\frac{D-3-p}{D-1-p}}\,, \tag{5.10}$$

where as before (see eq. (4.13)) we assume for convenience that the volume of the compactification to D dimensions is ~ 1 so that $\hat{\kappa} \sim g_s$. In the case that the $9-p$ transverse dimensions are isotropic, this is the same as eq. (5.9). The present case is (for convenience) taken to be one in which there is only one transverse dimension of size $R_\perp$ which is much larger than the string scale. We then set $D=5$ and $p=3$

and again find eq. (5.9). Thus our approximate solution for the particular case above breaks down precisely where the holographic bound is saturated - when the event horizon of a black-brane fills the transverse dimensions.

Note that, depending on the dimensionalities, the entire calculation (i.e. based on Hagedorn regime thermodynamics) can already break down at much lower densities corresponding merely to black brane dominance. Considering for definiteness black-hole type manifolds with planar asympotics, $S^1_\beta \times T^p_L$, it was seen in ref. [6] that when

$$\hat{\kappa}^2 \rho \gg 1 \tag{5.11}$$

the dominant component in the thermodynamics (i.e. that phase with the largest entropy) is the horizon of a Schwarzschild-like black brane. Since $R_\perp$ is larger than the string scale, eq. (5.11) can be satisfied at a density much lower than the density of the holographic bound above.

Adding in the constraint that the Yang-Mills entropy be less than the Hagedorn, we find that the phase diagram restrictions can be summarized as

$$\min\left\{R_\perp^{D-3-p}, 1\right\} \gg \hat{\kappa}^2\rho \gg \hat{\kappa}^2 N^2\,, \tag{5.12}$$

where N is the number of branes situated at $y = 0$. (The additional requirement of $\rho \gg \rho_c \sim 1$ explains why we need to be at a small string coupling, $\hat{\kappa}^2 \ll 1$.)

One consequence of the phase diagram constraints is that in the full solution a is not allowed to vary significantly in the range $y \in [-l, l]$, and hence, as promised, $\alpha_{||}(0) \approx 1$ in all the regions of validity. Note that a'' and n'' must still compensate for T_0^0 being greater than T_5^5 (the saddle-point dominance condition) and these two requirements are compatible only at weak coupling.

5.1.4 Stabilization and the second brane

A nice check of our approximate solution is that it should obey the stabilization condition discussed in [10, 14] for the case where $T_{05} = 0$ (as we are assuming throughout),

$$\int dy \sqrt{g_{55}}\left(T_0^0 + T_i^i - 2T_5^5\right) = 0\,. \tag{5.13}$$

To see that it indeed does without any further assumptions we need to consider what happens to the solution at the boundaries of the compact space. Assume that the large transverse dimension is compactified on a range $[-l, l]$ and recall that for convenience we are assuming Z_2 symmetry under $y \to -y$. Thus at $y = l$ the solutions have a discontinuity in the derivatives of a and n which is effectively a *second* brane. Using the discontinuity condition at $y = l$,

$$\begin{aligned} a'_{-l} - a'_{+l} &= -\frac{\hat{\kappa}^2}{3}\rho_{\rm br} a(l) b_0 \\ n'_{-l} - n'_{+l} &= +\frac{\hat{\kappa}^2}{3}(3p_{\rm br} + 2\rho_{\rm br}) n(l) b_0\,, \end{aligned} \tag{5.14}$$

we find that this corresponds to additional terms in the energy-momentum tensor given support at $y = l$ by a delta function;

$$\Delta T^0_0 = \delta(y-l)\rho_{\rm br} = \delta(y-l) R_\perp \left(\hat{\rho} - 3\frac{q^2}{\hat{\kappa}^2 t^2}\right)$$
$$\Delta T^i_i = \delta(y-l) p_{\rm br} = \delta(y-l) R_\perp \left(-\hat{p}_\gamma - \frac{3q^2-2q}{\hat{\kappa}^2 t^2}\right), \tag{5.15}$$

where $R_\perp = b_0 l$ and where we have neglected terms of order $\hat{\kappa}^4 \hat{\rho}^2 R_\perp^3$ in accord with our previous discussion. Note that the signs of the brane contributions are the opposite of those coming from the bulk, so that eq. (5.13) now reduces to

$$\int dy \left(\frac{2q^2-q}{t^2} + T^5_5\right) = 0\,. \tag{5.16}$$

This equation is automatically satisfied since it is the 55 component of Einstein's equations.

In the stabilized system (i.e. with the $\dot{b} = 0$ ansatz) the brane we find at $y = l$ inevitably has a peculiar equation of state that does not resemble a Yang-Mills or cosmological constant on the brane. Nevertheless the stabilization condition should always be satisfied so that, in a full model which includes a stabilization mechanism (given our assumptions for $a' = b' = n' = 0$ at $y = 0$ and Z_2 symmetry), the gravitational sector should behave *as if* there were a brane with these characteristics located at $y = l$.

5.2 L[γ] with $\frac{\hat{\kappa}^2}{12}\Lambda^2_{\rm br} - \Lambda_{\rm bulk} = 0$ and $\dot{b} \neq 0$

We now study the case where the extra dimension is not stabilized but there is still no nett cosmological constant. By 'nett' we mean that the contribution from a' and n' in G^5_5 cancels the bulk contribution on the r.h.s. of the T^5_5 equation. The condition for this is

$$\Lambda_{\rm nett} = -\frac{\hat{\kappa}^2}{12}\Lambda^2_{\rm br} + \Lambda_{\rm bulk} = 0\,. \tag{5.17}$$

In this case we could again, by studying the 00 and ii equations, try to find an approximate solution which is valid throughout the Hagedorn regime as we did in the previous subsection. However we would not learn anything further by doing this since we could always independently adjust $a_2(t)$ and $n_2(t)$ to satisfy the equations. (There is nothing other than the phase diagram constraints to predetermine these parameters.) In addition eq. (5.13) need not be satisfied (and in fact the integral is $\propto \dot{b}$), but one should keep in mind the fact that the full solution again has discontinuities in a' and n' at $y = l$ so that in general there should be a brane (or something that can compensate in a similar way) situated at $y = l$ even when $\dot{b} \neq 0$. The next subsection shows that by eliminating the second brane at $y = l$ one gains an extra constraint on a'' or n'' and hence useful additional information from the ii and 00 equations.

Therefore we will first focus on the 55 equation. Since $\alpha \approx 1$, for $\gamma = -1$, $-1/2$, $1/2$, the pressure is

$$\hat{p}_\gamma \propto a^{\frac{-3\gamma}{\gamma-1}} b^{-3/2} . \tag{5.18}$$

We will now catalogue some possible types of behaviour for a with different ansätze for b.

Taking $\Lambda_{\rm nett} = 0$, we find a family of power law solutions for $\gamma = -1$, $-1/2$, $1/2$ of the form

$$\begin{aligned} a_0(t) &\approx At^q \\ b_0(t) &\approx Bt^r , \end{aligned} \tag{5.19}$$

where subscript-0 indicates values at $y = 0$,

$$\begin{aligned} q &= \frac{\gamma-1}{2\gamma}\left(\frac{4}{3} - r\right) , \\ A^{\frac{3\gamma}{\gamma-1}} B^{3/2} &= \frac{\hat{\kappa}^2}{3q(2q-1)} , \end{aligned} \tag{5.20}$$

and where r is arbitrary. Note that if we choose $r = 4/3$ then $b \sim t^{4/3}$ and a is stationary whatever the value of γ (the complementary situation to b stationary and a undergoing power law inflation in eq. (5.2)).

We also find a family of hyperbolic solutions of the form

$$\begin{aligned} \frac{a_0(t)}{a_0(0)} &= \left(\frac{\sinh 2C(t+t_1)}{\sinh 2Ct_1}\right)^{1/2} \\ \frac{b_0(t)}{b_0(0)} &= \begin{cases} \left(\dfrac{\sinh 2C(t+t_1)}{\sinh 2Ct_1}\right)^{\frac{\gamma}{1-\gamma}} & \text{if } \gamma = -1, -\dfrac{1}{2}, \dfrac{1}{2} \\ \log\left(\dfrac{\sinh 2C(t+t_1)}{\sinh 2Ct_1}\right) & \text{if } \gamma = 0 \\ \exp\left(\dfrac{\sinh 2C(t+t_1)}{\sinh 2Ct_1}\right) & \text{if } \gamma = 1 . \end{cases} \end{aligned} \tag{5.21}$$

where C and t_1 are defined by

$$\begin{aligned} C^2 &= \frac{\hat{\kappa}^2}{6}\hat{p}_\gamma(0) \\ \frac{\dot{a}_0(0)}{a_0(0)} &= C \coth 2Ct_1 , \end{aligned} \tag{5.22}$$

$\hat{p}_\gamma$ is given by eq. (3.29), and where e.g. $\hat{p}_\gamma(0)$, $\rho(0) \equiv \hat{p}_\gamma(t=0)$, $\rho(t=0)$. Note that the solutions with $t_1 \to \pm\infty$ describe the scale factor exponentially increasing or decreasing; e.g. for $\gamma = -1, -1/2, 1/2$ in this limit we have

$$\begin{aligned} \frac{a_0(t)}{a_0(0)} &\approx \exp(\pm Ct) , && t_1 \to \pm\infty \\ \frac{b_0(t)}{b_0(0)} &\approx \exp\left(\pm\frac{2\gamma}{1-\gamma}Ct\right) , && t_1 \to \pm\infty . \end{aligned} \tag{5.23}$$

In the hyperbolic solutions T_5^5 is constant in time (see eq. (5.18). Both expansion and collapse of a are possible, since b compensates appropriately (in such a way as to keep T_5^5 constant). It seems rather paradoxical that the compensation can go either way depending on the sign of γ. For example, in the $\gamma = 1/2$ solution of eq. (5.23), if $a(t)$ is expanding then, in order to keep T_5^5 constant, b has to *expand* exponentially as well. The solution in the previous subsection for the $\dot{b} = 0$, $\gamma = 0$ system (section 5.1.1(b), eq. (5.4)) is similar, although there is a slight difference because there b is constant, resulting in the e^{-t^2} 'exit' term. We should also remember (bearing in mind our previous discussion) that *if adiabaticity is assumed* then the density T_0^0 is always diluted exponentially fast as $T_0^0 \propto \exp\left(-Ct^{\frac{3-\gamma}{1-\gamma}}\right)$. Finally we should add that which, if any, of these solutions is appropriate depends on the initial value we choose for $\dot{b}_0$.

5.3 $\mathbf{L}[\gamma]$ with $\frac{\hat{\kappa}^2}{12}\Lambda_{\rm br}^2 - \Lambda_{\rm bulk} \neq 0$ and $\dot{b} \neq 0$

Next we consider the case of additional cosmological constants in the brane and bulk. We transfer the a' and n' terms to the r.h.s. of the 55 equation and write

$$T^5_{5\,eff} = -\left(\hat{p}_\gamma + \Lambda_{\rm nett}\right), \tag{5.24}$$

where

$$\Lambda_{\rm nett} = \Lambda_{\rm bulk} - \frac{\hat{\kappa}^2}{12}\Lambda_{\rm br}^2 \,. \tag{5.25}$$

First we can see (rather trivially) that the hyperbolic solutions of the previous subsection in eq. (5.21) still exist if we modify C to C', where

$$C'^2 = \frac{\hat{\kappa}^2}{6}\left(\Lambda_{\rm nett} + \hat{p}_\gamma(0)\right), \tag{5.26}$$

and provided $C'^2 > 0$. Not surprisingly we recover the usual cosmological constant driven inflation when we set $\rho = 0$. When $C'^2 < 0$ (which is the case for $\Lambda_{\rm nett} < 0$ and $|\Lambda_{\rm nett}| > \hat{p}_\gamma(0)$), we find a singular collapsing solution for a;

$$\begin{aligned}
\frac{a_0(t)}{a_0(0)} &= \left(\frac{\sin 2|C'|(t+t_1)}{\sin 2|C'|t_1}\right)^{1/2} \\
\frac{b_0(t)}{b_0(0)} &= \begin{cases} \left(\dfrac{\sin 2|C'|(t+t_1)}{\sin 2|C'|t_1}\right)^{\frac{\gamma}{1-\gamma}} & \text{if } \gamma = -1, -\dfrac{1}{2}, \dfrac{1}{2} \\ \log\left(\dfrac{\sin 2|C'|(t+t_1)}{\sin 2|C'|t_1}\right) & \text{if } \gamma = 0 \\ \exp\left(\dfrac{\sin 2|C'|(t+t_1)}{\sin 2|C'|t_1}\right) & \text{if } \gamma = 1\,. \end{cases}
\end{aligned} \tag{5.27}$$

where t_1 is defined by

$$\frac{\dot{a}_0(0)}{a_0(0)} = |C'|\cot 2|C'|t_1\,. \tag{5.28}$$

For the power law solutions eq. (5.19), we note that the Hagedorn contribution to T_5^5 varies as $1/t^2$ and therefore decreases in time compared to the constant $\Lambda_{\rm nett}$ contributions. Thus we can consider the 55 equation in the two limits that $T^5_{5\,eff}$ in eq. (5.24) is dominated either by the cosmological constant term or by the Hagedorn term. For example, if $\gamma = -1, -1/2, 1/2$, we have

$$\left.\begin{aligned} a_0(t) &\approx A(t-t_0)^q \\ b_0(t) &\approx B(t-t_0)^r \end{aligned}\right\} \quad T^5_{5\,\rm eff} \approx -\hat{p}_\gamma \tag{5.29}$$

$$\left.\begin{aligned} \frac{a_0(t)}{a_0(0)} &= \left(\frac{\sinh 2C'(t+t_1)}{\sinh 2C' t_1}\right)^{1/2} \\ \frac{b_0(t)}{b_0(0)} &= \left(\frac{\sinh 2C'(t+t_1)}{\sinh 2C' t_1}\right)^{\frac{\gamma}{1-\gamma}} \end{aligned}\right\} \quad |T^5_{5\,\rm eff}| \approx |\Lambda_{\rm nett}| \gg |\hat{p}_\gamma| , \qquad C'^2 > 0 \tag{5.30}$$

$$\left.\begin{aligned} \frac{a_0(t)}{a_0(0)} &= \left(\frac{\sin 2|C'|(t+t_1)}{\sin 2|C'| t_1}\right)^{1/2} \\ \frac{b_0(t)}{b_0(0)} &= \left(\frac{\sin 2|C'|(t+t_1)}{\sin 2|C'| t_1}\right)^{\frac{\gamma}{1-\gamma}} \end{aligned}\right\} \quad |T^5_{5\,\rm eff}| \approx |\Lambda_{\rm nett}| \gg |\hat{p}_\gamma| , \qquad C'^2 < 0 \tag{5.31}$$

where t_0 is the time at which a is released from its small initial value.

5.4 Bouncing universe with softened singularity

If $\Lambda_{\rm nett} < 0$ the above solutions admit an oscillating universe. For example if $r < 4/3$, we find a universe in which a negative cosmological constant causes a to follow the sin curve of eq. (5.31) which reaches a maximum at $|C'|(t+t_1) = \pi/4$ before heading towards zero. However, unlike the case of pure negative cosmological constant in ordinary FRW, we do not hit a singularity because, when the scale factor a becomes small, $\Lambda_{\rm nett} + \hat{p}_\gamma$ in eq. (5.24) becomes positive and T_5^5 is dominated by $\hat{p}_\gamma$. Instead of hitting a singularity, a reaches some minimum value and rebounds upwards on the power law solution of eq. (5.29) (with t_0 now marking the time of the rebound, $\dot{a}(t_0, 0) = 0$). The scale factor a follows the power law curve until the Hagedorn contribution to T_5^5 (which varies as $1/(t-t_0)^2$) drops below the cosmological constant piece, at which point we pick up the sin curve of eq. (5.31) and have completed a single oscillation. The oscillation is around the completely static solution (with $\dot{a} = \ddot{a} = \dot{b} = 0$) where the pressure $\hat{p}_\gamma$ is exactly compensated by a negative cosmological constant,

$$\hat{p}_\gamma = -\Lambda_{\rm nett} \, . \tag{5.32}$$

The behaviour of b depends on the value of γ. For $\gamma = 1/2$ we find that b contracts and expands with a. In this case, at small a we must have a power law solution with $r > 4/3$ so that the contracting/expanding power law solution matches onto a contracting/expanding sin solution. On the other hand for $\gamma = -1$ or $-1/2$, b contracts and expands out of phase with a, and we require $0 < r < 4/3$ at small a. Clearly if, at small a, $r \approx 0$ then from eq. (5.29) b is almost static.

We stress that this singularity smoothing behaviour may or may not occur depending on the initial conditions (e.g. $\dot{b}$) and additional constraints on the system. For example, it does occur in numerical solutions with $\dot{b} = 0$.

It is encouraging to see that one contribution of stringy physics to the cosmology can be to soften the singularity that would otherwise appear at $a = 0$. This is a familiar aspect of string theory, but an appealing feature of the Hagedorn phase is that we can find it purely perturbatively.

5.5 Full solution with an example of a 'physical' brane

In the previous subsections we discussed the cosmological behaviour for various ansätze. However, apart from the $\dot{b} = 0$ constraint (which can be motivated by some unknown stabilization mechanism) the other ansätze are unmotivated, and we have still to show that the behaviour can arise in 'realistic' physical systems. The reason for this unwanted freedom is that the system of equations is underconstrained — there are four parameters (a_0, b_0, a_2, n_2) but only three independent equation (00, ii, 55). Therefore, we now consider an example in which we impose an additional constraint coming from a particular choice of equation of state on the brane. This allows us to find a full solution, in which the behaviour observed in the previous subsection arises in certain limits.

As we saw when we discussed stabilization, there is a discontinuity in a' and n' which corresponds to a brane located at $y = l$. Generally, the equation of state for this brane will look rather unrealistic (as indeed it does for the static $\dot{b} = 0$ solution) and we are forced to account for it by invoking unknown contributions from the gravitational sector. Our first assumption is therefore that there is *no* second brane (or rather, in view of the need to conserve Ramond-Ramond charge, that only the brane at $y = 0$ has any cosmological effect).

With this assumption, continuity at $y = l$ requires $a'_l = n'_l = 0$ and we must drop the approximation that the single brane at $y = 0$ (our would-be universe) is 'empty' (in the sense that $a'_0 = n'_0 = 0$) and consider its energy and momentum as well. The Israel conditions tell us that this energy density is given by

$$\begin{aligned} 3[a'/a]_0 &= -\hat{\kappa}^2 b_0 \rho_{\rm br} \\ 3[n'/n]_0 &= \hat{\kappa}^2 b_0 (2\rho_{\rm br} + 3p_{\rm br})\,. \end{aligned} \tag{5.33}$$

We now make an additional assumption about the equation of state of the brane at $y = 0$; we assume it is also Hagedorn-like, i.e. as well as the bulk $\mathbf{L}[\gamma]$ degrees of freedom there is an additional *localized* Hagedorn component with excitations near the brane. Even though we do not understand the thermodynamic behaviour of this subsystem, we can be fairly confident that its energy momentum tensor is like pressureless matter (as all the Hagedorn systems are to a first approximation — they all have $|\rho_{\rm br}| \gg |p_{\rm br}|$). Note that, to simplify this example, we will neglect the brane tension contribution which acts like $\Lambda_{\rm br}$.

For $p_{\rm br} \approx 0$ it is therefore sufficient to impose the relation

$$\frac{n'}{n} = -\frac{2a'}{a} \tag{5.34}$$

for all y. Integrated, this gives

$$n = \frac{a_0(t)^2}{a(t)^2}\,. \tag{5.35}$$

Together with $a'_l = n'_l = 0$ this means that the approximate solutions for $a(t,y)$ and $n(t,y)$ are going to be of the form

$$\begin{aligned} a(t,y) &= a_0(t) + a_1(t)\left(|y| - \frac{y^2}{2l}\right) \\ n(t,y) &= 1 - 2\frac{a_1(t)}{a_0(t)}\left(|y| - \frac{y^2}{2l}\right). \end{aligned} \tag{5.36}$$

The 05 equation may easily be solved and we find

$$b\,\lambda(y) = a'a^2\,, \tag{5.37}$$

where $\lambda(y)$ is a constant of integration. Note that this and eq. (5.33) imply that

$$-\frac{\hat\kappa^2}{3}\rho_{\rm br} = \frac{\lambda_0}{a_0(t)^3} \tag{5.38}$$

so that the brane density is indeed diluted by the expansion of the scale factor a like pressureless matter.

We now apply this to the 00, ii and 55 Einstein equations which now are more constrained by the continuity at $y = l$ and by our choice of equation of state. Somewhat fortuitously the condition in eq. (5.34) gives a cancellation of the a'' and n'' terms and the equations at $y = 0$ reduce to

$$\begin{aligned} \frac{\dot a_0}{a_0}\left(\frac{\dot a_0}{a_0} + \frac{\dot b_0}{b_0}\right) - \frac{\lambda_0^2}{a_0^6} - \frac{\lambda'_0}{a_0^3 b_0} &= \frac{\hat\kappa^2}{3}\hat\rho \\ 3\frac{\lambda_0^2}{a_0^6} + \frac{\dot a_0}{a_0}\left(\frac{\dot a_0}{a_0} + 2\frac{\dot b_0}{b_0}\right) + 2\frac{\ddot a_0}{a_0} + \frac{\ddot b_0}{b_0} &= -\hat\kappa^2\hat p_\gamma \\ \frac{\lambda_0^2}{a_0^6} + \left(\frac{\dot a_0}{a_0}\right)^2 + \frac{\ddot a_0}{a_0} &= +\frac{\hat\kappa^2}{3}\hat p_\gamma\,. \end{aligned} \tag{5.39}$$

Dominance of the $\hat p$ pressure term in the 55 equation is only possible if

$$\frac{\lambda_0^2}{a_0^6} \ll \left(\frac{\dot a}{a}\right)^2 \sim |\widehat{\kappa^2}\hat p_\gamma| \ll \widehat{\kappa^2}\hat\rho \tag{5.40}$$

and so the 00 equation must be dominated by the λ' term. Solving gives

$$\frac{\lambda(y)}{\lambda_0} - 1 \approx \frac{\rho}{\rho_{\rm br}}\frac{y}{l}\,. \tag{5.41}$$

Consistency of our solutions therefore requires that $\rho_{\rm br} \gtrsim \rho$ and hence by eq. (5.40)

$$\hat{\kappa}^2\rho \ll \frac{\hat{p}_\gamma}{\rho} \ll \frac{1}{V_\perp}\,. \tag{5.42}$$

As for the previous solutions, this corresponds to the holographic constraint in the phase diagram, and implies small coupling. Thus we again observe that, because only $\hat{\kappa}^4\rho_{\rm br}^2$ appears in the Einstein equations, the cosmology can be dominated by the 'bulk' energy-density *even when* $\rho < \rho_{\rm br}$, provided that the coupling is small.

For the $\mathbf{L}[-1]$ systems the ii and 55 equations may be solved by substituting $d = a_0(t)b_0(t)/a_0(0)b_0(0)$ and $c = a_0(t)^2/a_0(0)^2$ so that

$$\frac{\ddot{c}}{c} = \frac{2\hat{\kappa}^2}{3}\hat{p}_{-1}\,; \qquad \frac{\ddot{d}}{d} = -\frac{4\hat{\kappa}^2}{3}\hat{p}_{-1} = -\frac{4\hat{\kappa}^2}{3}\hat{p}_{-1}(0)d^{-3/2}\,. \tag{5.43}$$

Defining

$$w = \dot{d}_0(0)^2 - \frac{16\hat{\kappa}^2\hat{p}_{-1}(0)}{3}\sqrt{d_0(0)}$$

we find power law and hyperbolic behaviour emerging in different limits. When

$$t - t_0 \gg \frac{w^{3/2}}{\hat{\kappa}^4\hat{p}_{-1}(0)^2} \tag{5.44}$$

we have power law behaviour (eq. (5.19)) with

$$\left(\frac{4}{3} - r\right)\left(\frac{5}{3} - 2r\right) = \frac{1}{\hat{\kappa}^2\hat{p}_{-1}(0)}. \tag{5.45}$$

So for large $\hat{p}_{-1}$, $r \approx 4/3,\ 5/6$. On the other hand when

$$t - t_0 \ll \frac{w^{3/2}}{\hat{\kappa}^4\hat{p}_{-1}(0)^2} \tag{5.46}$$

we have $\sqrt{d} \approx -w/\hat{\kappa}^4\hat{p}_{-1}(0)^2 \approx \sqrt{d(0)}$ and hence we find the hyperbolic solution of eq. (5.30) with expanding a and collapsing b.

6. Sustaining inflation and solving the Horizon Problem

One of the most striking features of the cosmological solutions we have found is that they automatically predict a superluminal growth of the scale factor which is usually an ingredient of standard inflation. The remaining point that needs to be addressed is that our analysis assumed adiabaticity, and in order to realistically solve any cosmological problems, this assumption (as we will see) almost certainly requires modification. In this section we discuss solving the horizon problem beginning with the causality condition and then addressing the issue of adiabaticity. We then discuss how nonabiabatic effects may contribute to a sustained period of inflation.

6.1 Causality condition

An estimate of the size of the observable universe today is given by the distance light could travel between photon decoupling and now,

$$d_{\rm obs} \sim a(t_o) \int_{t_{\rm dec}}^{t_o} \frac{dt}{a(t)} \,. \tag{6.1}$$

Note that $d_{\rm obs} = \mathcal{O}(1) \times (t_o - t_{\rm dec})$ for $a \propto t^p$ and $p = \mathcal{O}(1)$ between $t_{\rm dec}$ and t_o. Here a is the scale factor. We can compare the comoving size of the observable universe to the comoving size of a causally connected region at some earlier time t_p:

$$\frac{d_{\rm hor}(t_p)}{a(t_p)} = \int_0^{t_p} \frac{dt}{a(t)} \,. \tag{6.2}$$

The observable universe today fits inside a causally connected region at t_p if

$$\frac{d_{\rm hor}(t_p)}{a(t_p)} \geq \frac{d_{\rm obs}}{a(t_o)} \,. \tag{6.3}$$

Here, subscript-o refers to today and subscript-p indicates some primordial time before the inflationary period of interest. If condition eq. (6.3) is met, then the horizon size at t_p (before nucleosynthesis) is large enough to allow for a causal explanation of the smoothness of the universe today. Note that more creative explanations of large scale smoothness may not involve comparing these two patches. For example, in the context of the brane scenarios, one might imagine that two regions of our observable universe which seem to be causally disconnected might in fact have talked to each other because of a geodesic between them that went off our brane, into the bulk, and then back onto our brane at some distant point, as demonstrated by Chung and Freese [18]. In the remainder of our discussion here we restrict ourselves to the case where eq. (6.3) is relevant; this is certainly the case for all the brane and boundary inflation models considered to date.

For power law expansion of the scale factor both before t_p and after $t_{\rm dec}$ (which may or may not be the case), we can take $t \sim H^{-1}$ during these periods. The causality condition eq. (6.3) then becomes

$$\frac{1}{a_p H_p} \geq \frac{1}{H_o a_o} \,, \tag{6.4}$$

or, equivalently,

$$\frac{t_p}{a_p} \geq \frac{t_{\rm today}}{a_{\rm today}} \,. \tag{6.5}$$

We take the Hubble constant today to be

$$H_o = \frac{\alpha_o^{1/2} T_o^2}{m_{pl}(t_o)} \,, \tag{6.6}$$

where $\alpha_o = (8\pi/3)(\pi^2/30)g_*(t_o)\eta_o$, g_* is the number of relativistic degrees of freedom and $\eta(t_o) \sim 10^4 - 10^5$ is the ratio today of the energy density in matter to that in radiation. From eq. (6.4) we can then see that accelerated expansion of the scale factor with $\ddot{a} > 0$ is required to solve the horizon problem. As we have shown throughout the paper , such accelerated expansion can easily occur during the Hagedorn regime.

We must here assume that, within an intitial horizon patch of size a_p, there has been enough smoothing that it is sensible to talk about a single value of the temperature and entropy density. This same assumption must be made in every inflation model.

6.2 The issue of non-adiabaticity

Adiabatic Hagedorn inflating systems can *in principle* already solve the horizon problem (as opposed to standard inflation in which nonadiabaticity is required). These systems could start with a large initial entropy due to the proximity of the initial temperature to the Hagedorn temperature. Subsequently, the entropy remains constant as the temperature drops and the scale factor a grows by a large amount. However *in practice* it is difficult to solve the horizon problem assuming adiabaticity: it is not possible to introduce *a priori* the entire entropy of the observable universe (i.e. $S \sim 10^{88}$) onto a primordial brane of order the string scale without either destroying the brane or having an unreasonably small string coupling (i.e. $g_s \sim 10^{-88}$!).

Consider for example the $\mathbf{L}[-1]$ system with $a \sim t^{4/3}$. Then from the first entry in Table I, we see that the entropy on a world-volume a_p^3 is

$$S_p \sim \beta_H a_p^3 \rho \sim \beta_H a_p^3 \frac{1}{(\beta - \beta_H)^2} \frac{1}{V_\perp}. \tag{6.7}$$

We will assume that the entropy on the 3-brane today is $S \propto (a_{\rm today} T_{\rm today})^3$.

We obtain the temperature/time relation during the Hagedorn phase from $\rho \sim a^{-3} \sim t^{-4}$ by equating this expression for ρ with that from the first entry in table 1. We find $t_p = V_\perp^{1/4}\sqrt{\beta_p - \beta_H}$. Eq. (6.3) can therefore be satisfied if

$$\frac{S_0}{S_p} > \frac{m_s}{T_{\rm today}} \left(V_\perp(\beta_p - \beta_H)^2\right)^{1/4} = \frac{m_s}{T_{\rm today}} \rho_p^{-1/4}. \tag{6.8}$$

Hence even an adiabatic system here can in principle solve the horizon problem: provided that the primordial density is large enough (or the primordial temperature sufficiently close to T_H) the horizon problem is solved even if $S_0 = S_p$. Indeed, we can see from eq. (6.7) that the initial entropy density can be as large as we like provided that we are prepared to accept a β that is arbitrarily close to β_H.

The serious practical difficulty arises however when we consider the stability bound in eq. (4.14) since it clearly implies that non-perturbative effects will be important unless the string coupling is fantastically weak. Indeed, generally if one assumes adiabaticity in attempting to solve the horizon problem, one requires the entire entropy of the observable universe to be present in string excitations at $t = t_p$. Hence, if the volumes are initially of order the string scale, then the energy density must be enormous, $\rho(t_p) \sim 10^{88}$ (e.g. in eq. (6.8)). After 60 e-folds of inflation $\rho(t)$ falls below ρ_c, the critical density which is needed to be in the Hagedorn phase, and the Yang-Mills phase takes over. However, for such a large initial value of the energy density, brane stability at $t = t_p$ (i.e. the constraint in eq. (4.14)) requires an initial value of $g_s < 10^{-88}$.

Because of this it is worth looking more critically at the supposed adiabaticity. This is a crucial assumption that almost certainly requires modification in a de Sitter background since the latter possesses a horizon with its own associated entropy. Hence, even if adiabaticity were the correct criterion (which it is not), it would only be meaningful for the coupled string/de Sitter system since there is a backreaction of the horizon on the string gas. The upshot for strings in a de Sitter background is that they are unstable to fluctuations and that this instability can sustain a period of de Sitter inflation [19, 20, 21]. This phenomenon is well known for strings once they are in the de Sitter-like phase.[4] However the missing ingredient that the present study adds is an explanation for how the universe enters a de Sitter like phase in the first place. Our mechanism gets the universe into the locally de Sitter phase, whereupon the mechanism of refs. [19, 20, 21] keeps it there *without* having to assume adiabaticity.

To achieve a sustained period of inflation therefore, we could just invoke the findings of refs. [19, 20, 21]. However for the remainder of this section we briefly elaborate on this property of strings, using purely heuristic arguments, in order to indicate a possible direction for future study.

Let us return to the density of states and consider instead a truly static metric in which the relevant Killing field is the time-translation,

$$K^\mu \equiv \frac{\partial}{\partial t} . \tag{6.9}$$

De Sitter space is an example of a static metric; in an appropriate coordinate system the metric is

$$ds^2 = -dt^2(1 - H^2r^2) + \frac{1}{1 - H^2r^2}dr^2 + r^2d\Omega^2 \tag{6.10}$$

and the Killing field has components $(1, 0, 0, 0)$. We can therefore discuss thermodynamics by compactifying on an imaginary time coordinate with $it \equiv it + \beta$.

A suitable modification to discuss the present case is to add an additional y coordinate with a metric $g_{55} = b(y)^2$ which is time independent in *this* coordinate system and also independent of r;

$$ds^2 = -(1 - H^2r^2)dt^2 + \frac{1}{1 - H^2r^2}dr^2 + r^2d\Omega^2 + b(y)^2dy^2 . \tag{6.11}$$

[4] We qualify 'de Sitter' because the exponential solutions we have do not possess the full O(1,4) de Sitter symmetry.

Under this assumption each 4 dimensional leaf of the foliated space is a de Sitter space and the geometry unambiguously fixes the Hawking temperature of the Horizon, $2\pi/H = \beta$.

It is well established (see for example ref. [22] and references therein) that the combined (geometric plus thermal) system obeys a generalized second law which is that the total entropy,

$$S = S_{\rm matter} + S_{\rm horizon} = S_{\rm matter} + \frac{1}{4\hat{\kappa}^2} A_{\rm horizon}\,, \tag{6.12}$$

obeys $\delta S > 0$, where $A_{\rm horizon}$ is the area of the horizon. The constant-time surface Σ extends to the horizon $r \approx 1/H = \beta/2\pi$ so that

$$V_{||} = {\rm Vol}(S_{d_{||}}) r^{d_{||}} \sim \beta^{d_{||}}; \qquad A_{\rm horizon} = {\rm Vol}(S_{d_{||}-1}) r^{d_{||}-1} \sim \beta^{d_{||}-1}\,, \tag{6.13}$$

where ${\rm Vol}(S_{d_{||}})$ is the volume of the unit sphere in $d_{||}$ dimensions.

Thus, assuming equilibrium at the temperature β^{-1}, the total entropy of an $\mathbf{L}[-1]$ string gas in a de Sitter background is of the form

$$S \sim \frac{\beta^{d_{||}}}{V_\perp (\beta - \beta_H)^2} + \frac{\beta^{d_{||}-1}}{4\pi\hat{\kappa}^2}\,. \tag{6.14}$$

This function has a minimum at

$$T_{\rm crit} = T_H \left(1 - \mathcal{O}\left((\hat{\kappa}^2/V_\perp)^{1/3}\right)\right). \tag{6.15}$$

Below $T_{\rm crit}$ fluctuations tend to drive the universe towards low temperature, i.e. out of the de Sitter phase.[5] The horizon has a negative specific heat which becomes relatively larger as the temperature drops so that equilibrium can be maintained (the condition being $|C_{V-}| > C_{V+}$). Above $T_{\rm crit}$ in the presence of a thermal gas of strings the flow is in the opposite direction towards an asymptotically limiting Hubble constant, $H_{\rm asymp} = 2\pi T_H$. Thus the generalized second law indicates that fluctuations in the geometry tend to drive the universe even further into the de Sitter phase provided that the Hubble constant is greater than the critical value. The entropy that the horizon loses when the Hubble constant is increased is more than offset by the huge increase in string entropy. Moreover, since the specific heat of the string gas becomes larger when the temperature increases, inevitably equilibrium is lost and energy flow to the strings becomes apparently limitless.

There is a slight difference between this picture and the results of ref. [20] which we should comment on. In ref. [20], the density never goes *above* a critical density, ρ_c', and indeed asymptotes to it from below as one goes back in time. In our case,

[5] We recognize that, in making this heuristic argument, we are on rather thin ice since we are simultaneously assuming a string coupling which is small enough for our approximations to be correct, but large enough to maintain equilibrium between the different degrees of freedom.

on the other hand, we must have $\rho > \rho_c \sim 1$ for the calculation to be valid (i.e. to be in the Hagedorn regime). Indeed in the present paper the energy density diverges as we approach the Hagedorn temperature. The difference is because ref. [20] was concerned with *stretched* strings and introduced a cut-off in the momentum integral in order to find the fractal dimension of the string. This 'coarse graining' put an artificial upper limit on the amount of string that can be packed into the volume. In fact they found the maximum fractal dimension (i.e. 2) only at the critical density ρ'_c. A fractal dimension of 2 is typical of the random walk behaviour which exists once Hagedorn behaviour sets in. So the coarse graining in ref. [20] effectively removed the Hagedorn behaviour and consequently ref. [20] found that $\rho = \rho'_c$ gave $\beta = \beta_H$. Conversely, the present paper begins in the regime $\rho > \rho_c$ where the calculations of ref. [20] end.

7. Conclusion and discussion

In this paper we have studied the possible cosmological implications of the Hagedorn regime of open strings on D-branes in the weak coupling limit. Our main result in sections 2–5 is that, due to the non-extensive dependence of the free energy on the volumes, a gas of open strings can exhibit negative pressure leading naturally to a period of power law or even exponential inflation — Hagedorn inflation. We also find that the open string gas can dominate the cosmological evolution at weak coupling even though the D-brane tension becomes large in this limit.

Hagedorn inflation also has a natural exit since any significant cooling can cause a change in the thermodynamics if winding modes become quenched or if the density drops below the critical density, $\rho_c \sim 1$, needed for the entropy of the Hagedorn phase to be dominant. Such a cooling can be caused by a sudden adiabatic increase in the transverse radius or by the inflation itself. We find this 'easy-exit' feature of open-string Hagedorn inflation to be of its most appealing features. In addition, we found that a small but negative cosmological constant, can cause the universe to enter a stable but oscillating phase. The effect of the Hagedorn phase is to soften the singular behaviour associated with the bounce.

The most striking aspect of our discussion is probably the existence of negative pressure. One might therefore ask how general a feature this is expected to be. By T-dualizing we argued that we can put the negative pressure down to the fact that in any particular direction, the gravitational degrees of freedom have both Kaluza-Klein modes and winding modes whereas the open strings (which are dominant in the entropy) have only one or the other. Hence we expect negative pressure to be possible whenever there are large space-filling modes that dominate the entropy.

In section 6 we speculated on how the inflation might be sustained through the well known phenomenon of string instability in a de Sitter background.

Hagedorn inflation may be thought of as a first example in the search for alternatives to the cosmological constant within the framework of string/brane systems. One possible direction for further study in this area is connected with the fact that we have throughout been taking the string coupling to be weak enough so that the brane tension does not play an important role in the cosmology. It is therefore interesting to ask if new cosmological effects might arise from large scale fluctuations in the branes themselves [6, 23]. The arguments of section 6 indicate that if it does then the brane driven inflation may be qualitatively different to the string driven inflation discussed here. This is because open strings are one dimensional objects with $S \sim E + \text{const.}\sqrt{E}$ whereas fluctuating p-branes have an entropy [24]

$$S \sim E^{\frac{2p}{p+1}} . \tag{7.1}$$

On calculating $\beta = \partial S/\partial E$ we see that $p \neq 1$ branes do not have the divergent behaviour which is the defining feature of strings. The thermodynamics of these objects has been the subject of much study [24] and there is probably more to be learnt here.

Finally, an interesting connected issue which we did not discuss is related to the effect of brane melting discussed in ref. [6], in which the non-perturbative aspect of D-brane thermal production must be taken into account. At the present time it is hard to make any quantitative estimate of these effects on the cosmological solutions we have been discussing here, but we hope to be able to address this question in future work.

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